Mechanics of Deformable Porous Media

Finite Strains, Slot 1: Kinematics, Hyperelasticity, and Dissipation

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Linear elasticity assumed strains are infinitesimal, which is often violated. Soft porous media, gels, membranes, and biological tissue undergo finite strain, or large solid-body motion that linear elasticity falsely characterizes as deformation. This lecture formulates kinematics and elasticity without the small-strain assumption, and briefly outlines the extension to plasticity theory and phase-field modeling of fracture. We focus on quasi-static loading, with t a loading parameter.

1. Configurations and deformation map

We begin by focusing on the kinematics of deformation and motion:

Definition: Motion and configurations

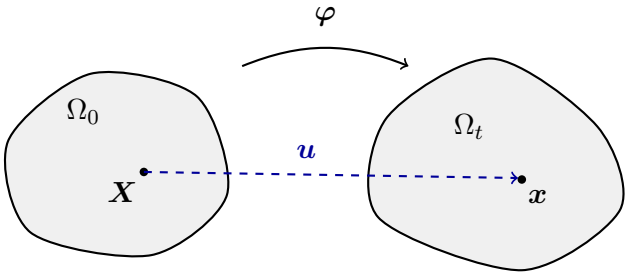
Given a *reference* configuration Ω_0, a motion maps each material point $\boldsymbol{X}$ to a new $\boldsymbol{x}$:

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t), \qquad \Omega_t := \boldsymbol{\varphi}(\Omega_0, t), \tag{1}$$

with Ω_t denoting the *current* configuration. The displacement is defined as:

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{\varphi}(\boldsymbol{X}, t) - \boldsymbol{X}. \tag{2}$$

Note $\boldsymbol{X}$ labels material points, and $\boldsymbol{x}$ its current position.



Fields can be written as functions of $\boldsymbol{X}$, the *material description*, or of $\boldsymbol{x}$, the *spatial description*. Computations in solid mechanics are more tractable in the material description.

2. The deformation gradient and strain

All local measures of deformation derive from the gradient of motion map.

Definition: Deformation gradient

The deformation gradient is defined as follows:

$$\boldsymbol{F} := \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} = \mathbf{1} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}. \tag{3}$$

which maps line segments via $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$, and volumes via $dv = J\, dV$, where $J := \det \boldsymbol{F} > 0$. Note $J = 0$ and $J < 0$ are unphysical as they either destroy material or turn it inside out.

$\boldsymbol{F}$ contains both strain and rigid rotation. We next separate the former from the latter.

Definition: Cauchy–Green tensor and Green–Lagrange strain

To filter out rotation, use $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$ to compute the segment length:

$$|d\boldsymbol{x}|^2 = d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{X}, \qquad \boldsymbol{C} := \boldsymbol{F}^\top \boldsymbol{F}, \tag{4}$$

The tensor $\boldsymbol{C}$ is the *right Cauchy–Green tensor*, symmetric and positive definite. The Green–Lagrange strain measures the change in the length of this line segment:

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = 2\, d\boldsymbol{X} \cdot \boldsymbol{E}\, d\boldsymbol{X}, \qquad \boldsymbol{E} := \tfrac{1}{2}(\boldsymbol{C} - \mathbf{1}) = \tfrac{1}{2}\Big(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^\top + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}^\top \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\Big). \tag{5}$$

If displacement gradient is small, the quadratic term is negligible and $\boldsymbol{E} \approx \boldsymbol{\varepsilon}$.

Proposition: Polar decomposition

The deformation gradient splits uniquely into a rotation $\boldsymbol{R}$ following a stretch $\boldsymbol{U}$:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}, \qquad \boldsymbol{U} := \sqrt{\boldsymbol{C}}. \tag{6}$$

Consequently $\boldsymbol{C} = \boldsymbol{U}\boldsymbol{R}^\top \boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^2$. Hence, $\boldsymbol{C}$ and $\boldsymbol{E}$ depend on stretch alone.

Proof. $\boldsymbol{C}$ is symmetric positive definite, so its square root $\boldsymbol{U}$ exists, is unique and invertible. Define $\boldsymbol{R} := \boldsymbol{F}\boldsymbol{U}^{-1}$. Then $\boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{U}^{-1}\boldsymbol{F}^\top \boldsymbol{F}\boldsymbol{U}^{-1} = \boldsymbol{U}^{-1}\boldsymbol{U}^2\boldsymbol{U}^{-1} = \mathbf{1}$, and $\det \boldsymbol{R} = J/\det \boldsymbol{U} > 0$, so $\boldsymbol{R}$ is a rotation. □

Example: Rigid rotation

Consider a solid-body rotation $\boldsymbol{\varphi} = \boldsymbol{Q}\boldsymbol{X}$. Hence, $\boldsymbol{F} = \boldsymbol{Q}$, $\boldsymbol{C} = \boldsymbol{Q}^\top \boldsymbol{Q} = \mathbf{1}$, and $\boldsymbol{E} = \mathbf{0}$, implying no deformation. Small strain $\boldsymbol{\varepsilon}$ mistakes this rotation for deformation, if rotation

is large. To see this, write the displacement $\boldsymbol{u} = (\boldsymbol{Q} - \mathbf{1})\boldsymbol{X}$ and compute $\partial \boldsymbol{u}/\partial \boldsymbol{X} = \boldsymbol{Q} - \mathbf{1}$. Now $\boldsymbol{\varepsilon} = \operatorname{sym}(\boldsymbol{Q} - \mathbf{1}) = \frac{1}{2}(\boldsymbol{Q} + \boldsymbol{Q}^\top) - \mathbf{1} \neq 0$. For example, a 2D rotation with angle θ:

$$\boldsymbol{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \Longrightarrow \quad \tfrac{1}{2}(\boldsymbol{Q} + \boldsymbol{Q}^\top) = \cos\theta\, \mathbf{1} \quad \Longrightarrow \quad \boldsymbol{\varepsilon} = (\cos\theta - 1)\, \mathbf{1}, \tag{7}$$

Note $\boldsymbol{\varepsilon} \neq 0$ records a spurious volumetric compression. For small angles, $\cos\theta - 1 \approx -\theta^2/2$ and the error is second order, which is acceptable for the linear theory.

3. Stress measures

Forces act in the current configuration, but our description is material. Therefore, stress can be expressed in more than one form depending on which area it references.

Definition: Cauchy stress

The Cauchy stress $\boldsymbol{\sigma}$ lives on the deformed body. The traction on the current surface with normal $\boldsymbol{n}$ is $\boldsymbol{t} = \boldsymbol{\sigma}\, \boldsymbol{n}$. It is symmetric and corresponds to *true stress*, what a gauge measures.

Referring traction to the reference area requires knowing how area elements deform.

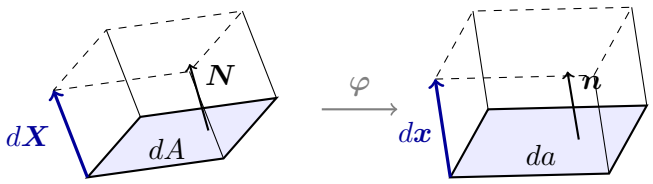
Proposition: Nanson's formula

Oriented area elements transform as:

$$\boldsymbol{n}\, da = J\, \boldsymbol{F}^{-\top}\, \boldsymbol{N}\, dA, \tag{8}$$

with $\boldsymbol{N}\, dA$ denoting the reference element and $\boldsymbol{n}\, da$ its deformed image.

Proof. Consider the prism spanned by the area and a line segment, transformed as $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$.



The corresponding volumes are $dV = d\boldsymbol{X} \cdot \boldsymbol{N}\, dA$ and $dv = d\boldsymbol{x} \cdot \boldsymbol{n}\, da$, related by $dv = J\, dV$. Thus:

$$d\boldsymbol{x} \cdot \boldsymbol{n}\, da = J\; d\boldsymbol{X} \cdot \boldsymbol{N}\, dA. \tag{9}$$

Substituting $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$ and moving $\boldsymbol{F}$ across the inner product, $(\boldsymbol{F}\, d\boldsymbol{X}) \cdot \boldsymbol{n} = d\boldsymbol{X} \cdot (\boldsymbol{F}^\top \boldsymbol{n})$:

$$d\boldsymbol{X} \cdot \left(\boldsymbol{F}^\top \boldsymbol{n}\, da\right) = d\boldsymbol{X} \cdot \left(J\, \boldsymbol{N}\, dA\right) \qquad \forall\, d\boldsymbol{X}, \tag{10}$$

Since the above is valid for all $d\boldsymbol{X}$, the claim is proven. □

Definition: Piola stresses

The current force per unit reference area is the *first Piola–Kirchhoff stress*:

$$\boldsymbol{t}\, da = \boldsymbol{\sigma}\, \boldsymbol{n}\, da = \underbrace{J\, \boldsymbol{\sigma}\, \boldsymbol{F}^{-\top}}_{=:\, \boldsymbol{P}}\, \boldsymbol{N}\, dA, \tag{11}$$

Note $\boldsymbol{P}$ is not symmetric. Pulling the force vector itself back to the reference configuration defines the *second Piola–Kirchhoff stress*:

$$\boldsymbol{S} := \boldsymbol{F}^{-1}\, \boldsymbol{P} = J\, \boldsymbol{F}^{-1} \boldsymbol{\sigma}\, \boldsymbol{F}^{-\top}. \tag{12}$$

Note $\boldsymbol{S}$ is symmetric and measures reference force per unit reference area.

Example. Consider a bar is pulled axially by a force f at its ends, with cross-sectional area A_0 in the reference configuration and a after deformation. The axial component $P_{11} = f/A_0$ is the *engineering stress*, and $\sigma_{11} = f/a$ is the *true stress*. For small strains, the two coincide.

Proposition: Work conjugacy

The current stress power per unit reference volume is:

$$\boldsymbol{P} : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{E}}. \tag{13}$$

Recall $\boldsymbol{P}$ is current force per reference area and $\dot{\boldsymbol{F}}$ is current length per reference length.

Proof. Using $\boldsymbol{P} = \boldsymbol{F}\, \boldsymbol{S}$ and the trace identity $\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}^\top \boldsymbol{B})$, we can write:

$$\boldsymbol{P} : \dot{\boldsymbol{F}} = \operatorname{tr}\left(\boldsymbol{S}^\top \boldsymbol{F}^\top \dot{\boldsymbol{F}}\right) = \boldsymbol{S} : \left(\boldsymbol{F}^\top \dot{\boldsymbol{F}}\right). \tag{14}$$

Since $\boldsymbol{S}$ is symmetric, only the symmetric part of $\boldsymbol{F}^\top \dot{\boldsymbol{F}}$ survives, which is precisely $\dot{\boldsymbol{E}} = \frac{1}{2}(\dot{\boldsymbol{F}}^\top \boldsymbol{F} + \boldsymbol{F}^\top \dot{\boldsymbol{F}}) = \operatorname{sym}(\boldsymbol{F}^\top \dot{\boldsymbol{F}})$, obtained by differentiating $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{1})$. Hence $\boldsymbol{S} : (\boldsymbol{F}^\top \dot{\boldsymbol{F}}) = \boldsymbol{S} : \dot{\boldsymbol{E}}$. □

Note $(\boldsymbol{P}, \dot{\boldsymbol{F}})$ and $(\boldsymbol{S}, \dot{\boldsymbol{E}})$ are conjugate pairs, and either can be used to formulate free energy and dissipation. Below, we use $(\boldsymbol{S}, \dot{\boldsymbol{E}})$, since $\boldsymbol{S}$ is symmetric and $\boldsymbol{E}$ measures true deformation.

4. Hyperelasticity and frame indifference

The constitutive theory postulates a free energy per unit reference volume:

Definition: Hyperelastic material

A material is hyperelastic if free energy depends on deformation only, $\psi = \psi(\boldsymbol{F})$. Thus:

$$\mathcal{D} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi} = \left(\boldsymbol{P} - \frac{\partial \psi}{\partial \boldsymbol{F}}\right) : \dot{\boldsymbol{F}} \ \geq \ 0, \tag{15}$$

By the Coleman–Noll argument, the following holds:

$$\boldsymbol{P} = \frac{\partial \psi}{\partial \boldsymbol{F}}, \qquad \mathcal{D} = 0. \tag{16}$$

As in the small-strain case, energy is stored without dissipation.

Axiom: Frame indifference

The stored energy cannot change under rigid *rotation* of the deformed state. To characterize this rotation with a mapping from the reference state, we can write $d\boldsymbol{X} \to \boldsymbol{F}\, d\boldsymbol{X} \to \boldsymbol{Q}\,\boldsymbol{F}\, d\boldsymbol{X}$. The composite deformation gradient is therefore $\boldsymbol{QF}$. Frame indifference demands:

$$\psi(\boldsymbol{Q}\,\boldsymbol{F}) = \psi(\boldsymbol{F}) \qquad \forall\ \boldsymbol{Q}. \tag{17}$$

Proposition: Reduced form of free energy

Frame indifference is equivalent to free energy being a function of $\boldsymbol{C}$ alone:

$$\psi(\boldsymbol{F}) = \hat{\psi}(\boldsymbol{C}). \tag{18}$$

Proof. If $\psi = \hat{\psi}(\boldsymbol{C})$, replacing $\boldsymbol{F}$ by $\boldsymbol{QF}$ leaves $\boldsymbol{C}$ unchanged, $(\boldsymbol{QF})^{\top}(\boldsymbol{QF}) = \boldsymbol{F}^{\top}\boldsymbol{Q}^{\top}\boldsymbol{Q}\,\boldsymbol{F} = \boldsymbol{C}$. Therefore $\psi(\boldsymbol{QF}) = \hat{\psi}(\boldsymbol{C}) = \psi(\boldsymbol{F})$, and frame indifference holds. Conversely, if $\psi(\boldsymbol{QF}) = \psi(\boldsymbol{F})$ for all rotations $\boldsymbol{Q}$, then compute the polar decomposition $\boldsymbol{F} = \boldsymbol{RU}$ and set $\boldsymbol{Q} = \boldsymbol{R}^{\top}$. Hence, $\psi(\boldsymbol{F}) = \psi(\boldsymbol{R}^{\top}\boldsymbol{F}) = \psi(\boldsymbol{U})$, and because $\boldsymbol{U} = \sqrt{\boldsymbol{C}}$ we prove the claim $\psi(\boldsymbol{F}) = \hat{\psi}(\boldsymbol{C})$. □

Proposition: Piola stresses from reduced energy

Piola stresses derives from the reduced energy as follows:

$$\boldsymbol{S} = 2\,\frac{\partial \hat{\psi}}{\partial \boldsymbol{C}}, \qquad \boldsymbol{P} = \boldsymbol{F}\,\boldsymbol{S}. \tag{19}$$

Proof. First, apply the chain rule to $\psi = \hat{\psi}(\boldsymbol{C}(t))$ to obtain:

$$\dot{\psi} = \frac{\partial \hat{\psi}}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}} = 2\,\frac{\partial \hat{\psi}}{\partial \boldsymbol{C}} : \dot{\boldsymbol{E}}. \tag{20}$$

where we used $\dot{\boldsymbol{C}} = \dot{\boldsymbol{F}}^\top \boldsymbol{F} + \boldsymbol{F}^\top \dot{\boldsymbol{F}} = 2\dot{\boldsymbol{E}}$, obtained by differentiating $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F} = 2\boldsymbol{E} + \boldsymbol{1}$. Separately,

$$\dot{\psi} = \boldsymbol{P} : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{E}} \tag{21}$$

where the first equality comes from zero dissipation $\mathcal{D} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi} = 0$ and the second from work conjugacy. Equating the two expressions for $\dot{\psi}$ yields:

$$\left(\boldsymbol{S} - 2\,\frac{\partial \hat{\psi}}{\partial \boldsymbol{C}}\right) : \dot{\boldsymbol{E}} = 0 \qquad \forall\ \dot{\boldsymbol{E}}. \tag{22}$$

which proves the claim $\boldsymbol{S} = 2\,\partial\hat{\psi}/\partial\boldsymbol{C}$. □

For isotropic materials the energy is additionally unchanged by rotations of the reference configuration and depends on $\boldsymbol{C}$ only through its invariants. In practice two invariants suffice, $I_1 = \operatorname{tr}\boldsymbol{C}$ and $J = \det\boldsymbol{F} = \sqrt{\det\boldsymbol{C}}$. The latter equality comes from $\det\boldsymbol{C} = \det(\boldsymbol{F}^\top\boldsymbol{F}) = (\det\boldsymbol{F})^2$.

Remark: Derivative of tensor determinant

If $\boldsymbol{C}$ is invertible and symmetric, and $\delta\boldsymbol{C}$ a small perturbation:

$$\det(\boldsymbol{C} + \delta\boldsymbol{C}) = \det\boldsymbol{C}\ \det\left(\boldsymbol{1} + \boldsymbol{C}^{-1}\delta\boldsymbol{C}\right) = \det\boldsymbol{C}\left(1 + \operatorname{tr}(\boldsymbol{C}^{-1}\delta\boldsymbol{C})\right) + O(\delta^2), \tag{23}$$

The last equality holds because the eigenvalues of $\boldsymbol{1} + \boldsymbol{A}$, where $\boldsymbol{A}$ is a small tensor, are $1 + a_i$. Thus, $\det(\boldsymbol{1} + \boldsymbol{A}) = \prod_i (1 + a_i) = 1 + \sum_i a_i + O(\boldsymbol{A}^2) = 1 + \operatorname{tr}\boldsymbol{A} + O(\boldsymbol{A}^2)$. Moreover:

$$\frac{\partial \det\boldsymbol{C}}{\partial\boldsymbol{C}} = (\det\boldsymbol{C})\,\boldsymbol{C}^{-\top} = (\det\boldsymbol{C})\,\boldsymbol{C}^{-1} \quad\Longrightarrow\quad \frac{\partial \ln J}{\partial\boldsymbol{C}} = \frac{1}{2}\,\frac{\partial \ln\det\boldsymbol{C}}{\partial\boldsymbol{C}} = \frac{1}{2}\,\boldsymbol{C}^{-1}. \tag{24}$$

Example: Neo-Hookean material

A compressible Neo-Hookean solid is defined by:

$$\psi = \tfrac{\mu}{2}\,(I_1 - 3) - \mu\,\ln J + \tfrac{\lambda}{2}\,(\ln J)^2, \tag{25}$$

with λ, μ constants. Using $\partial I_1/\partial\boldsymbol{C} = \boldsymbol{1}$ and $\partial \ln J/\partial\boldsymbol{C} = \frac{1}{2}\,\boldsymbol{C}^{-1}$, from the above remark:

$$\boldsymbol{S} = 2\,\frac{\partial\psi}{\partial\boldsymbol{C}} = \mu\,(\boldsymbol{1} - \boldsymbol{C}^{-1}) + \lambda\,(\ln J)\,\boldsymbol{C}^{-1}. \tag{26}$$

If there is no deformation, $\boldsymbol{C} = \boldsymbol{1}$ and $J = 1$, and stress vanishes as it must.

Proposition: Linearization recovers linear elasticity

For small strains, the Neo-Hookean model reduces to Hooke's law:

$$\boldsymbol{S} \approx \lambda\,(\operatorname{tr}\boldsymbol{\varepsilon})\,\boldsymbol{1} + 2\mu\,\boldsymbol{\varepsilon}, \tag{27}$$

with λ, μ the Lamé constants, and equality valid to first order.

Proof. If $\boldsymbol{E} \approx \boldsymbol{\varepsilon}$, then $\boldsymbol{C} = \mathbf{1} + 2\boldsymbol{E} \approx \mathbf{1} + 2\boldsymbol{\varepsilon}$. Hence, $\boldsymbol{C}^{-1} \approx \mathbf{1} - 2\boldsymbol{\varepsilon}$ because $(\mathbf{1} + 2\boldsymbol{\varepsilon})(\mathbf{1} - 2\boldsymbol{\varepsilon}) = \mathbf{1} - 4\boldsymbol{\varepsilon}^2 \approx \mathbf{1}$. Moreover, we can approximate $\det \boldsymbol{C} \approx \det(\mathbf{1} + 2\boldsymbol{\varepsilon}) \approx 1 + \operatorname{tr}(2\boldsymbol{\varepsilon})$, followed by $\ln J = \frac{1}{2} \ln \det \boldsymbol{C} \approx \frac{1}{2} \operatorname{tr}(2\boldsymbol{\varepsilon}) = \operatorname{tr} \boldsymbol{\varepsilon}$, using $\ln(1 + x) \approx x$. Substituting into Neo-Hookean stress:

$$\boldsymbol{S} \approx \mu\big(\mathbf{1} - (\mathbf{1} - 2\boldsymbol{\varepsilon})\big) + \lambda\,(\operatorname{tr}\boldsymbol{\varepsilon})(\mathbf{1} - 2\boldsymbol{\varepsilon}) = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,(\operatorname{tr}\boldsymbol{\varepsilon})\,\mathbf{1} - 2\lambda\,(\operatorname{tr}\boldsymbol{\varepsilon})\,\boldsymbol{\varepsilon}, \tag{28}$$

The last term is second-order, so we drop it and the claim is proved. □

Moreover, with $\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\,\boldsymbol{S}\,\boldsymbol{F}^{\top}$, $\boldsymbol{F} = \mathbf{1} + \partial \boldsymbol{u}/\partial \boldsymbol{X} \approx \mathbf{1}$, and $J \approx 1$, we get $\boldsymbol{S} \approx \boldsymbol{\sigma}$ up to first order.

5. Outlook: inelasticity at finite strain

The internal-variable machinery of the earlier lectures extends over to finite-strain mechanics. This section outlines two such extensions without attempting a detailed development.

Definition: Multiplicative split

Finite-strain plasticity replaces additive strain split by the multiplicative split:

$$\boldsymbol{F} = \boldsymbol{F}^e\,\boldsymbol{F}^p, \tag{29}$$

where $\boldsymbol{F}^p$ captures plastic deformation and $\boldsymbol{F}^e$ captures elastic deformation. The free energy depends on the elastic part only, through $\boldsymbol{C}^e := \boldsymbol{F}^{e\top}\boldsymbol{F}^e$, as follows:

$$\psi = \psi^e(\boldsymbol{C}^e) + \psi^h(\alpha). \tag{30}$$

Yield function, flow rule, and KKT conditions retain the same structure. At small strains, the additive form is recovered. Writing $\boldsymbol{F}^e = \mathbf{1} + \boldsymbol{H}^e$ and $\boldsymbol{F}^p = \mathbf{1} + \boldsymbol{H}^p$ with small $\boldsymbol{H}^e, \boldsymbol{H}^p$:

$$\boldsymbol{F} = \mathbf{1} + \boldsymbol{H}^e + \boldsymbol{H}^p + \boldsymbol{H}^e\boldsymbol{H}^p \approx \mathbf{1} + \boldsymbol{H}^e + \boldsymbol{H}^p, \tag{31}$$

where $\partial \boldsymbol{u}/\partial \boldsymbol{X} = \boldsymbol{H}^e + \boldsymbol{H}^p$. Taking the symmetric part yields $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$.

Definition: Dissipation inequality

Ignoring thermal effects, the dissipation inequality per unit reference volume reads:

$$\mathcal{D} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi} = \boldsymbol{S} : \dot{\boldsymbol{E}} - \dot{\psi} \;\geq\; 0, \tag{32}$$

where the second equality follows from work conjugacy.

Example: Viscoelastic solid

Consider a strain-rate dependent solid. The free energy and its rate are:

$$\psi = \psi(\boldsymbol{C}, \dot{\boldsymbol{C}}) \quad \Longrightarrow \quad \dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}} + \frac{\partial \psi}{\partial \dot{\boldsymbol{C}}} : \ddot{\boldsymbol{C}}. \tag{33}$$

Substituting into the dissipation inequality, and using $\dot{\boldsymbol{C}} = 2\dot{\boldsymbol{E}}$:

$$\mathcal{D} = \left(\boldsymbol{S} - 2\frac{\partial \psi}{\partial \boldsymbol{C}}\right) : \dot{\boldsymbol{E}} \; - \; \frac{\partial \psi}{\partial \dot{\boldsymbol{C}}} : \ddot{\boldsymbol{C}} \; \geq \; 0. \tag{34}$$

By the Coleman–Noll argument, the second term vanishes and the first defines viscous stress:

$$\frac{\partial \psi}{\partial \dot{\boldsymbol{C}}} = \boldsymbol{0}, \qquad \boldsymbol{S} = 2\frac{\partial \psi}{\partial \boldsymbol{C}} + \boldsymbol{S}^v, \tag{35}$$

Hence, the dissipation inequality reduces to:

$$\mathcal{D} = \boldsymbol{S}^v : \dot{\boldsymbol{E}} \; \geq \; 0. \tag{36}$$

The Kelvin–Voigt model derives $\boldsymbol{S}^v$ from the dissipation potential $\phi(\dot{\boldsymbol{C}}) = \frac{\eta}{2}\, \dot{\boldsymbol{C}} : \dot{\boldsymbol{C}}$:

$$\boldsymbol{S}^v = \frac{\partial \phi}{\partial \dot{\boldsymbol{C}}} = \eta\, \dot{\boldsymbol{C}} = 2\eta\, \dot{\boldsymbol{E}} \quad \Longrightarrow \quad \mathcal{D} = 2\eta\, \dot{\boldsymbol{E}} : \dot{\boldsymbol{E}} \; \geq \; 0. \tag{37}$$

Example: Plasticity at finite strain

Consider the following ingredients for rate-independent plasticity at finite strain:

$$\boldsymbol{F} = \boldsymbol{F}^e\, \boldsymbol{F}^p, \qquad \psi = \psi^e(\boldsymbol{C}^e) + \psi^h(\alpha), \qquad \boldsymbol{C}^e := \boldsymbol{F}^{e\top} \boldsymbol{F}^e, \qquad \boldsymbol{S}^e := 2\frac{\partial \psi^e}{\partial \boldsymbol{C}^e}. \tag{38}$$

We first compute $\boldsymbol{P} : \dot{\boldsymbol{F}}$ in the dissipation inequality:

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{F}}^e \boldsymbol{F}^p + \boldsymbol{F}^e \dot{\boldsymbol{F}}^p \quad \Longrightarrow \quad \boldsymbol{P} : \dot{\boldsymbol{F}} = \left(\boldsymbol{P}\, \boldsymbol{F}^{p\top}\right) : \dot{\boldsymbol{F}}^e + \left(\boldsymbol{F}^{e\top} \boldsymbol{P}\right) : \dot{\boldsymbol{F}}^p. \tag{39}$$

Next we differentiate the free energy by the chain rule, with $A := -\partial \psi^h / \partial \alpha$:

$$\dot{\psi} = \frac{\partial \psi^e}{\partial \boldsymbol{C}^e} : \dot{\boldsymbol{C}}^e + \frac{\partial \psi^h}{\partial \alpha}\, \dot{\alpha} = \tfrac{1}{2}\, \boldsymbol{S}^e : \dot{\boldsymbol{C}}^e - A\, \dot{\alpha}. \tag{40}$$

Since $\boldsymbol{S}^e$ is symmetric, we can replace $\dot{\boldsymbol{C}}^e = \dot{\boldsymbol{F}}^{e\top} \boldsymbol{F}^e + \boldsymbol{F}^{e\top} \dot{\boldsymbol{F}}^e$ by its non-symmetric form:

$$\tfrac{1}{2}\, \boldsymbol{S}^e : \dot{\boldsymbol{C}}^e = \boldsymbol{S}^e : \left(\boldsymbol{F}^{e\top} \dot{\boldsymbol{F}}^e\right) = \left(\boldsymbol{F}^e \boldsymbol{S}^e\right) : \dot{\boldsymbol{F}}^e. \tag{41}$$

Substituting into the dissipation inequality:

$$\mathcal{D} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\psi} = \left(\boldsymbol{P}\, \boldsymbol{F}^{p\top} - \boldsymbol{F}^e \boldsymbol{S}^e\right) : \dot{\boldsymbol{F}}^e \; + \; \left(\boldsymbol{F}^{e\top} \boldsymbol{P}\right) : \dot{\boldsymbol{F}}^p \; + \; A\, \dot{\alpha} \; \geq \; 0. \tag{42}$$

By the Coleman–Noll argument, $\dot{\boldsymbol{F}}^e$ is arbitrary so its coefficient vanishes:

$$\boldsymbol{P} = \boldsymbol{F}^e\, \boldsymbol{S}^e\, \boldsymbol{F}^{p-\top}. \tag{43}$$

Substituting back into the middle term and shuffling the transpose:

$$\left(\boldsymbol{F}^{e\top} \boldsymbol{P}\right) : \dot{\boldsymbol{F}}^p = \left(\boldsymbol{C}^e \boldsymbol{S}^e\, \boldsymbol{F}^{p-\top}\right) : \dot{\boldsymbol{F}}^p = \left(\boldsymbol{C}^e \boldsymbol{S}^e\right) : \left(\dot{\boldsymbol{F}}^p \boldsymbol{F}^{p-1}\right) =: \boldsymbol{M} : \boldsymbol{L}^p, \tag{44}$$

This defines the Mandel stress $\boldsymbol{M} := \boldsymbol{C}^e \boldsymbol{S}^e$ and plastic rate $\boldsymbol{L}^p := \dot{\boldsymbol{F}}^p \boldsymbol{F}^{p-1}$, such that:

$$\mathcal{D} = \boldsymbol{M} : \boldsymbol{L}^p + A\,\dot{\alpha} \;\geq\; 0. \tag{45}$$

Yield and flow then follow from the plasticity lecture with $(\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}^p)$ replaced by $(\boldsymbol{M}, \boldsymbol{L}^p)$:

$$f(\boldsymbol{M}, A) \leq 0, \qquad \boldsymbol{L}^p = \dot{\lambda}\,\frac{\partial f}{\partial \boldsymbol{M}}, \qquad \dot{\alpha} = \dot{\lambda}\,\frac{\partial f}{\partial A}, \qquad \dot{\lambda} \geq 0, \quad \dot{\lambda} f = 0. \tag{46}$$

Definition: Phase-field fracture at finite strain

The phase-field functional of the previous lecture extends to finite strain as follows:

$$\Lambda[\boldsymbol{u}, d] = \int_{\Omega_0} g(d)\,\psi_0(\boldsymbol{C})\;\mathrm{d}V + G_c \int_{\Omega_0} \gamma(d, \nabla d)\;\mathrm{d}V - \int_{\Omega_0} \boldsymbol{b}\cdot\boldsymbol{u}\;\mathrm{d}V - \int_{\Gamma_t} \boldsymbol{t}\cdot\boldsymbol{u}\;\mathrm{d}A. \tag{47}$$

Energy split is subtler at finite strain, since volumetric and deviatoric responses are no longer additive. As an example, let's consider the volumetric–isochoric split:

$$\bar{\boldsymbol{C}} := J^{-2/3}\,\boldsymbol{C} \quad\Longrightarrow\quad \det \bar{\boldsymbol{C}} = J^{-2} \det \boldsymbol{C} = 1, \tag{48}$$

where the pair $(J, \bar{\boldsymbol{C}})$ replaces $\boldsymbol{C}$—with J capturing volume change and $\bar{\boldsymbol{C}}$ capturing shape change (isochoric). Recall $\det \boldsymbol{C} = (\det \boldsymbol{F})^2 = J^2$. Free energy is then split additively:

$$\psi_0(\boldsymbol{C}) = \psi_{vol}(J) + \psi_{iso}(\bar{\boldsymbol{C}}), \tag{49}$$

and degradation acts on shape change, and expanding volume change:

$$\psi(\boldsymbol{C}, d) = \begin{cases} g(d)\,\big[\psi_{iso}(\bar{\boldsymbol{C}}) + \psi_{vol}(J)\big], & J \geq 1 \quad \text{(expansion)}, \\ g(d)\,\psi_{iso}(\bar{\boldsymbol{C}}) + \psi_{vol}(J), & J < 1 \quad \text{(compaction)}. \end{cases} \tag{50}$$

The driving energy in $\mathcal{H}$ is thus $\psi_0^+ = \psi_{iso} + \psi_{vol}$ if $J \geq 1$, and $\psi_0^+ = \psi_{iso}$ if $J < 1$.