Mechanics of Deformable Porous Media

Plasticity and Fracture, Slot 2: Phase-Field Fracture

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This lecture applies the variational and thermodynamic machinery of the course to fracture.

1. Griffith's view of fracture

Fracture creates new surface, and surface costs energy. Griffith (1921) viewed crack growth as an energy balance: elastic energy released versus surface energy spent. For simplicity, we consider quasi-static loading, isothermal and dry conditions, with no thermal or pore-fluid contributions.

Definition: Potential energy and energy release rate

Let $\Pi(A)$ be the total potential energy evaluated at equilibrium displacement, with the crack advancing along a presumed path so that the crack set is a one-parameter family $\Gamma(A)$:

$$\Pi(A) = \int_{\Omega\setminus\Gamma(A)} \psi(\boldsymbol{\varepsilon})\,\mathrm{d}V - \int_{\Omega} \boldsymbol{b}\cdot\boldsymbol{u}\,\mathrm{d}V - \int_{\Gamma_t} \boldsymbol{t}\cdot\boldsymbol{u}\,\mathrm{d}A. \tag{1}$$

The *energy release rate* is defined as:

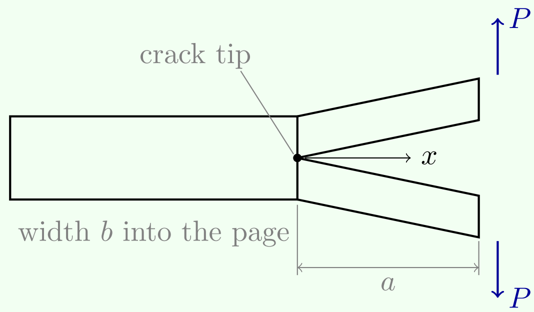
$$G := -\frac{d\Pi}{dA}, \tag{2}$$

with units of energy per area. G depends on geometry, loading, and elastic moduli through the equilibrium solution. It is a computed quantity, not a material constant.

Example. With all boundaries clamped, the load integral terms drop and $G = -dU/dA$, where U is the stored elastic energy. This is like cutting a stretched membrane, which releases elastic energy.

Example: The double cantilever beam

A beam of out-of-plane width b is split open by a crack of length a along its midplane, leaving two arms pulled apart by equal and opposite force P:



Each arm is a beam clamped at the crack tip and loaded at its free end. The beam theory of the earlier lecture gives the deflection $(EI\,w'')'' = 0$. The solution w is a cubic, with clamped conditions $w(0) = w'(0) = 0$ removing two constants, and the free-end conditions, zero moment $EI\,w''(a) = 0$ and prescribed shear, $-EI\,w'''(a) = P$, removing the rest:

$$w(x) = \frac{P}{6EI}\big(3a\,x^2 - x^3\big), \qquad \delta := w(a) = \frac{Pa^3}{3EI}. \tag{3}$$

The two free ends separate by $u = 2\delta$, so the structure is linear with compliance C:

$$u = C(A)\,P, \qquad C = \frac{2a^3}{3EI}, \qquad A = b\,a. \tag{4}$$

Load control. With only point loads, recalling the Castigliano and reciprocity lecture, $U = \frac{1}{2}\sum_i P_i u_i = \frac{1}{2}Pu$ for a linear structure and $W_{ext} = Pu$. Hence:

$$\Pi = U - W_{ext} = \tfrac{1}{2}Pu - Pu = -\tfrac{1}{2}\,C(A)\,P^2 \quad \Longrightarrow \quad \boxed{G = -\frac{d\Pi}{dA} = \frac{P^2}{2b}\frac{dC}{da} = \frac{P^2a^2}{b\,EI}} \tag{5}$$

Notice G vanishes as $a \to 0$, grows with crack at fixed load, and grows with load at fixed crack. The crack grows when G reaches toughness G_c (defined below). As the crack advances at fixed load the structure softens, $dC > 0$, the load does work $dW_{ext} = P\,du = P^2dC$, the stored energy rises by $dU = \frac{1}{2}P^2dC$, and the difference $-d\Pi = \frac{1}{2}P^2dC = G\,dA$ is released.

Displacement control. Eliminating the load through $P = u/C$ gives:

$$\boxed{G = \frac{9\,EI\,u^2}{4\,b\,a^4}} \tag{6}$$

now falling with crack length. Therefore, growth under *load control is unstable*, as G rises with a, whereas growth under *displacement control is stable*, as G falls with a. To see nucleation never occurs via Griffith, consider fixed load and let $a \to 0$. This yields $G \to 0$, which is always below G_c. Under fixed displacement, this argument does not apply because G blows up as $a \to 0$. But fortunately, that is *not* nucleation. Holding a finite opening across a vanishing crack requires $P = u/C \to \infty$, an infinite pre-crack state (unphysical).

Definition: Griffith criterion

Griffith found that quasi-static (stable) crack growth obeys:

$$\dot{A} \geq 0, \qquad G - G_c \leq 0, \qquad (G - G_c)\,\dot{A} = 0 \tag{7}$$

where G_c is the fracture toughness. If $G < G_c$, crack does not grow. If $G = G_c$, crack growth is stable. And if $G > G_c$, crack growth is dynamic (unstable), which is beyond this lecture.

These are KKT conditions. Similar to plasticity, $\dot{A}$ can be viewed as a flux, positive because cracks do not heal, and G as the conjugate force, bounded by G_c. Thus, fracturing is rate-independent with a threshold, similar to plasticity, and it satisfies the dissipation inequality $\mathcal{D} = G\,\dot{A} \geq 0$.

Griffith's criterion is best viewed as a constrained minimization of a total energy functional Λ:

$$\min_{A \geq A_{old}} \; \Lambda(A), \qquad \Lambda(A) = \Pi(A) + G_c\,A \tag{8}$$

whose KKT conditions are given above. The second condition is:

$$\frac{d\Lambda}{dA} = -G + G_c \geq 0 \tag{9}$$

which falls out naturally from optimality, implying cracks are either frozen or growing stably. To generalize this to arbitrary cracks Γ of un-presumed path, the total energy can be defined as:

$$\Lambda[\boldsymbol{u}, \Gamma] = \int_{\Omega \setminus \Gamma} \psi(\boldsymbol{\varepsilon})\;\mathrm{d}V + G_c\,|\Gamma| - \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{u}\;\mathrm{d}V - \int_{\Gamma_t} \boldsymbol{t} \cdot \boldsymbol{u}\;\mathrm{d}A. \tag{10}$$

and crack evolution is formulated as the joint constrained minimization:

$$\min_{\boldsymbol{u},\;\Gamma \supseteq \Gamma_{old}} \Lambda[\boldsymbol{u}, \Gamma] \;=\; \min_{\Gamma \supseteq \Gamma_{old}} \Big[\min_{\boldsymbol{u}} \; \Lambda[\boldsymbol{u}, \Gamma] \Big], \tag{11}$$

The inner minimization is over $\boldsymbol{u}$ and is unconstrained. Stationarity gives equilibrium at fixed crack. The outer minimization is over the crack set and seeks a *global* minimum, which is what permits nucleation. If crack path is known, $\Gamma = \Gamma(A)$ with $A = |\Gamma|$, constraint becomes $A \geq A_{old}$, and the problem reduces to minimizing $\Lambda(A) := \Lambda[\boldsymbol{u}_{eq}(A), \Gamma(A)]$. For a *local* minimum, setting:

$$\frac{d\Lambda}{dA} \;=\; \underbrace{\delta_{\boldsymbol{u}}\Lambda}_{=0 \text{ at equilibrium}} \cdot \frac{d\boldsymbol{u}_{eq}}{dA} \;+\; \frac{\partial \Lambda}{\partial A}\bigg|_{\boldsymbol{u}\text{ fixed}} \;\geq\; 0, \tag{12}$$

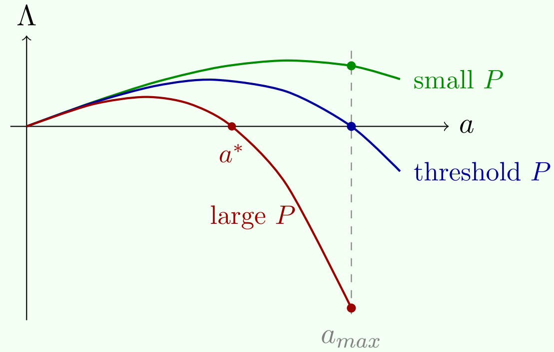
yields the KKT conditions of Griffith. This local search is what precludes nucleation, since $G \to 0$ as $A \to 0$, as shown in the example. In reality, nucleation is associated with an energy barrier that a local minimum misses. To see this, consider the cantilever example below:

Example: Crack nucleation in double cantilever beam

Load control. Under fixed load P:

$$\Lambda(a) = -\frac{P^2 a^3}{3EI} + G_c\, b\, a = a\left(G_c\, b - \frac{P^2 a^2}{3EI}\right), \tag{13}$$

where $\Lambda(0) = 0$. The following plots Λ versus load P:



The cost of new surface grows linearly with a then drops as a cubic, crossing zero at:

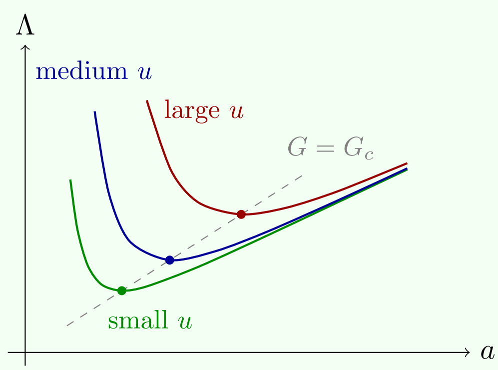
$$a^* = \frac{\sqrt{3\, EI\, b\, G_c}}{P}. \tag{14}$$

Crack sizes are bound by Ω's dimension, $a \leq a_{max}$. At small P, $\Lambda > 0$ for all a. Hence, the global minimum occurs at $a = 0$, and no crack forms. Increasing P decreases a^*, and past a threshold $P > \sqrt{3EI\, b\, G_c}/a_{max}$, the global minimum occurs at $a = a_{max}$. This corresponds to nucleation and complete rupture of Ω. Thus, under load control, crack growth is never stable. Local search near $a = 0$ would only see $d\Lambda/dA|_{A=0} = G_c > 0$ and decide no cracks form at any P. Notice $d\Lambda/da = 0$ locates the maximum of Λ and yields $G = G_c$. If the initial crack were located there, the slightest increase in load causes the crack to run to a_{max}.

Displacement control. Under fixed opening u of the cantilever lips, the picture is different:

$$\Lambda(a) = \frac{3EI\, u^2}{4\, a^3} + G_c\, b\, a, \qquad \frac{d\Lambda}{da} = b\left(G_c - G\right), \qquad G = \frac{9EI\, u^2}{4\, b\, a^4}, \tag{15}$$

with G now decreasing in a. The following shows a plot of Λ versus a:



The landscape is a valley with global minimum at $G = G_c$. Setting $d\Lambda/da = 0$, we get:

$$a_{eq} = \left(\frac{9\,EI\,u^2}{4\,b\,G_c}\right)^{1/4} \propto \sqrt{u}, \tag{16}$$

for equilibrium crack length. As u grows, so does a_{eq} continuously, i.e., stable growth. The intact state never minimizes Λ as opening is fixed. So the analysis is not useful for nucleation.

While convenient, the generalized $\Lambda[\boldsymbol{u}, \Gamma]$ poses three obstacles: (1) the minimization search space over all admissible Γ is formidable; (2) the surface measure $|\Gamma|$ requires explicit knowledge of crack geometry; (3) tracking discrete surfaces numerically is hard. The phase-field model addresses these.

2. Regularization: the phase field model

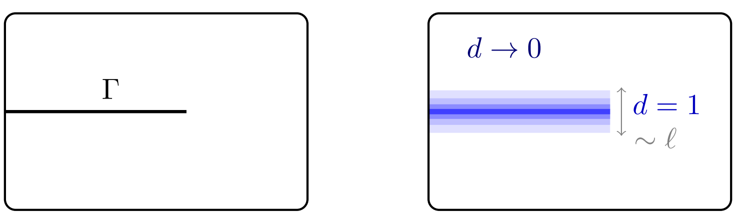
The phase-field model makes the minimization of $\Lambda[\boldsymbol{u}, \Gamma]$ tractable by regularizing Γ with a continuous damage variable over Ω. The key ingredient is the crack density function defined below.

Definition: Phase field and crack surface density

The phase field $d(\boldsymbol{x}) \in [0, 1]$ marks intact material at $d = 0$ and fully broken material at $d = 1$. The crack surface is replaced by a crack surface density $\gamma(d, \nabla d)$, such that:

$$|\Gamma| \;\approx\; \int_\Omega \gamma(d, \nabla d)\;\mathrm{d}V, \qquad \gamma(d, \nabla d) = \frac{d^2}{2\ell} + \frac{\ell}{2}\,|\nabla d|^2, \tag{17}$$

where $\ell > 0$ is the length scale determining the width of the diffuse crack.



sharp crack diffuse crack

The first term in γ penalizes damage, keeping d near zero away from cracks, while the second term penalizes sharp changes, spreading the transition over a width $O(\ell)$. As $\ell \to 0$ the diffuse crack converges to the sharp one. We prove the integral of γ equals $|\Gamma|$ later.

Remark: AT1 and AT2

The crack density above is one member of the Ambrosio–Tortorelli (AT) family:

$$\gamma = \frac{1}{4c_w}\Big(\frac{w(d)}{\ell} + \ell\,|\nabla d|^2\Big), \qquad c_w := \int_0^1 \sqrt{w(s)}\,ds. \tag{18}$$

AT2 takes $w = d^2$, giving $c_w = \frac{1}{2}$ and the density above. AT1 takes $w = d$, with $c_w = 2/3$, whose linear term produces an elastic threshold. This means that no damage occurs below a critical load with AT1, whereas AT2 damages at any load. We adopt AT2 throughout.

Definition: Degradation function

Damage weakens the material through a degradation function $g(d)$:

$$\psi(\boldsymbol{\varepsilon}, d) = g(d)\,\psi_0(\boldsymbol{\varepsilon}), \qquad \psi_0(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \quad \Rightarrow \quad \boldsymbol{\sigma} = \partial\psi/\partial\boldsymbol{\varepsilon} = g(d)\,\mathbb{C} : \boldsymbol{\varepsilon}, \tag{19}$$

with $g(0) = 1$ for intact and $g(1) = 0$ for fully-damaged material. Moreover, $g' \leq 0$ since stiffness never grows with damage. The standard choice is $g(d) = (1-d)^2$.

3. Phase-field functional and its Euler–Lagrange equations

Regularization turns Λ into a functional of two fields on the body's volume:

Definition: Phase-field functional

The regularized total energy functional is:

$$\Lambda[\boldsymbol{u}, d] = \int_\Omega g(d)\,\psi_0(\boldsymbol{\varepsilon}(\boldsymbol{u}))\;\mathrm{d}V + G_c\int_\Omega \gamma(d, \nabla d)\;\mathrm{d}V - \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}\;\mathrm{d}V - \int_{\Gamma_t} \boldsymbol{t}\cdot\boldsymbol{u}\;\mathrm{d}A. \tag{20}$$

where arguments are fields on Ω. This simplifies minimization compared to Γ in $\Lambda[\boldsymbol{u}, \Gamma]$.

Theorem: Phase-field Euler–Lagrange equations

Stationarity of $\Lambda[\boldsymbol{u}, d]$ with respect to $\boldsymbol{u}$ and d, with $\nabla d\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, yields:

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \;\text{ in } \Omega, \qquad \boldsymbol{\sigma} = g(d)\,\mathbb{C} : \boldsymbol{\varepsilon}, \qquad \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t} \;\text{ on } \Gamma_t, \tag{21}$$

$$\frac{G_c}{\ell}\big(d - \ell^2\nabla^2 d\big) = -g'(d)\,\psi_0 \;\text{ in } \Omega. \tag{22}$$

Proof. We first consider variation in $\boldsymbol{u}$. Perturb $\boldsymbol{u}$ by $\epsilon\,\boldsymbol{\eta}$ with $\boldsymbol{\eta} = \boldsymbol{0}$ on the displacement boundary. Since $\partial\psi/\partial\boldsymbol{\varepsilon} = g(d)\,\mathbb{C} : \boldsymbol{\varepsilon} =: \boldsymbol{\sigma}$, the first variation reads:

$$\delta_{\boldsymbol{u}}\Lambda = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\eta})\;\mathrm{d}V - \int_\Omega \boldsymbol{b}\cdot\boldsymbol{\eta}\;\mathrm{d}V - \int_{\Gamma_t} \boldsymbol{t}\cdot\boldsymbol{\eta}\;\mathrm{d}A = 0. \tag{23}$$

This is the virtual work statement of prior lectures with degraded stress. The usual integration by parts yields $\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$ in Ω and $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}$ on Γ_t.

We next consider variation in d. Perturb d by $\epsilon\, q$ with q arbitrary. The first variation is:

$$\delta_d\Lambda = \int_\Omega \Big[g'(d)\,\psi_0\, q + G_c\Big(\frac{d}{\ell}\, q + \ell\,\nabla d\cdot\nabla q\Big)\Big]\ \mathrm{d}V = 0. \tag{24}$$

For the gradient term, apply $\nabla\cdot(q\,\nabla d) = \nabla q\cdot\nabla d + q\,\nabla^2 d$ followed by the divergence theorem:

$$\int_\Omega \nabla d\cdot\nabla q\ \mathrm{d}V = \int_{\partial\Omega} q\,(\nabla d\cdot\boldsymbol{n})\ \mathrm{d}A - \int_\Omega q\,\nabla^2 d\ \mathrm{d}V, \tag{25}$$

where the boundary term vanishes by $\nabla d\cdot\boldsymbol{n} = 0$. Collecting terms, we get:

$$\int_\Omega \Big[g'(d)\,\psi_0 + G_c\Big(\frac{d}{\ell} - \ell\,\nabla^2 d\Big)\Big]\, q\ \mathrm{d}V = 0 \tag{26}$$

and the fundamental lemma gives the damage evolution equation. □

The second equation balances the crack driving force $-g'(d)\,\psi_0$ against the resistance $G_c(d/\ell - \ell\nabla^2 d)$. As written, stationarity treats growth and healing the same, fixed in later sections.

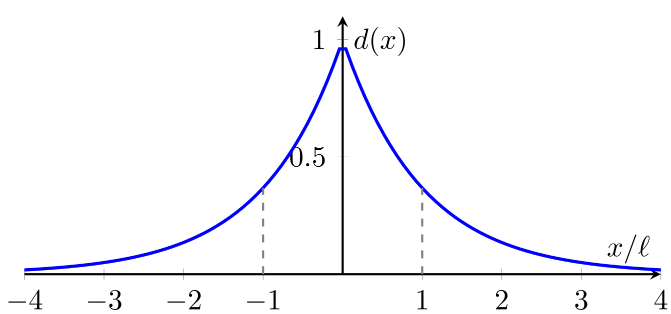
Proposition: The regularized crack area

In an unloaded body containing a straight, fully developed crack, the damage field decays from the crack plane as $d = e^{-|x|/\ell}$, with x the distance to the crack. Moreover,

$$A = |\Gamma| = \int_\Omega \gamma(d, \nabla d)\ \mathrm{d}V \tag{27}$$

Proof. With no load, $\psi_0 = 0$, the damage equation reduces to $d - \ell^2 d'' = 0$, with $d(0) = 1$ on the crack plane and decay at infinity. The decaying solution is $d = e^{-x/\ell}$ for $x \geq 0$, mirrored for $x < 0$. Substituting into γ then integrating on $x \in (-\infty, \infty)$ we get 1 per unit crack area.

$$\int_\Omega \gamma\ \mathrm{d}V = \int_\Gamma \Big[\underbrace{\int_{-\infty}^{\infty}\gamma\ \mathrm{d}x}_{=1}\Big]\ \mathrm{d}A = \int_\Gamma 1\ \mathrm{d}A = |\Gamma|. \qquad \square$$



4. History variable and the energy split

The Euler–Lagrange equations as derived do not prevent cracks from healing or forming under compressive loading. The following offers constitutive fixes to these pathologies.

Definition: History variable

One way to forbid healing is through a *history variable*:

$$\mathcal{H}(\boldsymbol{x}, t) := \max_{s \leq t} \ \psi_0\big(\boldsymbol{\varepsilon}(\boldsymbol{x}, s)\big), \tag{28}$$

replacing ψ_0 in the damage equation:

$$\frac{G_c}{\ell}\big(d - \ell^2 \nabla^2 d\big) = 2(1-d)\,\mathcal{H}. \tag{29}$$

Since $\mathcal{H}$ never decreases, d is non-decreasing and cracks never heal. During loading $\mathcal{H} = \psi_0$.

Definition: Energy split

Compression should not cause damage. Therefore, degradation due to damage acts on certain parts of the stored energy, which is split as follows:

$$\psi(\boldsymbol{\varepsilon}, d) = g(d)\,\psi_0^+(\boldsymbol{\varepsilon}) + \psi_0^-(\boldsymbol{\varepsilon}), \qquad \boldsymbol{\sigma} = g(d)\,\frac{\partial \psi_0^+}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \psi_0^-}{\partial \boldsymbol{\varepsilon}}. \tag{30}$$

The resulting driving energy in the definition of $\mathcal{H}$ becomes ψ_0^+. Two splits are standard.

Definition: Volumetric–deviatoric split

This split was introduced by Amor, Marigo, and Maurini (2009). Let $\langle x \rangle_\pm := (x \pm |x|)/2$ denote the Macaulay brackets, ε_v the volumetric strain, and $\boldsymbol{e}$ the deviatoric strain. Thus:

$$\psi_0^+ = \tfrac{1}{2}\,K\,\langle \varepsilon_v \rangle_+^2 + \mu\ \boldsymbol{e} : \boldsymbol{e}, \qquad \psi_0^- = \tfrac{1}{2}\,K\,\langle \varepsilon_v \rangle_-^2. \tag{31}$$

The volumetric-tensile and deviatoric parts degrade, while volumetric-compressive does not.

Definition: Spectral split

This split was introduced by Miehe, Welschinger, and Hofacker (2010). Let $\boldsymbol{\varepsilon} = \sum_i \varepsilon_i\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$ denote the spectral decomposition of strain and $\boldsymbol{\varepsilon}_\pm := \sum_i \langle \varepsilon_i \rangle_\pm\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$. Thus:

$$\psi_0^\pm = \tfrac{1}{2}\,\lambda\,\langle \operatorname{tr} \boldsymbol{\varepsilon} \rangle_\pm^2 + \mu\ \operatorname{tr}\left(\boldsymbol{\varepsilon}_\pm^2\right). \tag{32}$$

Tension along each principal direction degrades, while compression does not.

Remark: Choosing a split

The above splits satisfy $\psi_0^+ + \psi_0^- = \psi_0$ and make stress a nonlinear function of strain, even at fixed d. The volumetric–deviatoric split is the simpler and computationally cheaper of the two. The spectral split requires spectral decomposition at every point, which is costlier.

Summary: The phase-field fracture model

Summarizing, the equations solved in practice:

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}, \qquad \boldsymbol{\sigma} = (1-d)^2\,\frac{\partial \psi_0^+}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \psi_0^-}{\partial \boldsymbol{\varepsilon}}, \tag{33}$$

$$\frac{G_c}{\ell}\big(d - \ell^2\nabla^2 d\big) = 2(1-d)\,\mathcal{H}, \qquad \mathcal{H} = \max_{s\le t}\ \psi_0^+, \qquad \nabla d\cdot\boldsymbol{n} = 0. \tag{34}$$

5. Thermodynamics: fracture as an internal variable

The damage field can be viewed as an internal variable in the sense of the earlier lectures. However, the machinery there does not fully apply because of the gradient ∇d. Below we show a workaround.

Definition: Variational derivative

Given the first variation of a functional $I[d]$:

$$\delta I = \int_\Omega (\star)\ \delta d\ \mathrm{d}V \tag{35}$$

for every admissible perturbation δd, the variational derivative is $\delta I/\delta d := \star$.

Example. For $I = \int_\Omega F(d,\nabla d)\ \mathrm{d}V$ with $\nabla d\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, we get $\delta I/\delta d = \partial F/\partial d - \nabla\cdot\big(\partial F/\partial\nabla d\big)$.

Definition: KKT conditions of phase field

Define the driving force and resistance of fracture by:

$$Y := -\frac{\partial\psi}{\partial d} = -g'(d)\,\psi_0 = 2(1-d)\,\psi_0, \qquad R := G_c\,\frac{\delta}{\delta d}\int_\Omega \gamma\ \mathrm{d}V = \frac{G_c}{\ell}\big(d - \ell^2\nabla^2 d\big). \tag{36}$$

Phase field evolves through the KKT conditions:

$$\dot d \ge 0, \qquad Y - R \le 0, \qquad \dot d\,(Y-R) = 0. \tag{37}$$

denoting irreversibility, admissibility, complementarity. Notice $\delta\Lambda/\delta d = R - Y$. During growth, $Y = R$, which is the Euler–Lagrange equation. If $Y < R$, damage remains frozen.

The global pair $(G, \dot{A})$ from before is now the local $(Y, \dot{d})$ at each point, a repackaging of Griffith. Unlike the sharp crack criterion, nucleation is not precluded. In an intact loaded body, $d = 0$ and $Y - R = 2\psi_0 > 0$, an inadmissible state. Thus, damage must nucleate for KKT conditions to hold.

Proposition: Fracture and the dissipation inequality

Along the evolution path above, dissipation is locally nonnegative, and globally satisfies:

$$\mathcal{D} = Y\,\dot{d} \;\geq\; 0, \qquad \int_\Omega \mathcal{D}\;\mathrm{d}V = G_c\,\dot{A}, \qquad A = \int_\Omega \gamma\;\mathrm{d}V. \tag{38}$$

Proof. Since $\psi_0 \geq 0$ and $d \leq 1$, we have $Y = 2(1-d)\,\psi_0 \geq 0$. And $\dot{d} \geq 0$, so $\mathcal{D} \geq 0$. For the global statement, where $\dot{d} > 0$ complementarity gives $Y = R$, and where $\dot{d} = 0$ nothing contributes, so:

$$\int_\Omega \mathcal{D}\;\mathrm{d}V = \int_\Omega R\,\dot{d}\;\mathrm{d}V = G_c \int_\Omega \Big(\frac{d}{\ell}\,\dot{d} - \ell\,\nabla^2 d\;\dot{d}\Big)\;\mathrm{d}V. \tag{39}$$

Integrating the Laplacian term by parts with $\nabla d \cdot \boldsymbol{n} = 0$ turns $-\ell\,\nabla^2 d\;\dot{d}$ into $\ell\,\nabla d \cdot \nabla\dot{d}$. Since ∇ and the time derivative commute, the integrand becomes the time derivative of γ by chain rule:

$$\int_\Omega \mathcal{D}\;\mathrm{d}V = G_c \int_\Omega \Big(\underbrace{\frac{d}{\ell}\,\dot{d}}_{\frac{\partial\gamma}{\partial d}\,\dot{d}} + \underbrace{\ell\,\nabla d \cdot \nabla\dot{d}}_{\frac{\partial\gamma}{\partial\nabla d}\cdot\nabla\dot{d}}\Big)\;\mathrm{d}V = G_c\,\frac{d}{dt}\int_\Omega \gamma\;\mathrm{d}V = G_c\,\dot{A}. \qquad \square \tag{40}$$

Remark: Microforces

Strictly speaking, since ∇d is not a local internal variable, the correct thermodynamic framework is Gurtin's theory. A scalar microforce, conjugate to $\dot{d}$ and specified below, and a vector microstress, $G_c\,\ell\,\nabla d$, obey their own balance law. The dissipation inequality dictates their constitutive form, and the same equations as here emerge. Determining the force conjugate to $\dot{d}$ requires the derivative $\partial\varphi/\partial\dot{d}$ of a dissipation potential: $\varphi(\dot{d}) = R\,\dot{d}$ for $\dot{d} \geq 0$ and $+\infty$ for $\dot{d} < 0$. This is a one-sided cone with its corner located at zero flux, similar to plasticity.