Mechanics of Deformable Porous Media

Plasticity and Fracture, Slot 1: Rate-Independent Plasticity

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This lecture builds rate-independent plasticity from the thermodynamic framework.

1. The elastoplastic framework

A metal or soil loaded past a threshold incurs permanent deformation. Moreover, the measured stress–strain curve is nearly independent of the imposed strain rate. Plasticity is the theory built on these two observations. From the previous lecture, the isothermal dissipation inequality is,

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} \;\geq\; 0, \tag{1}$$

and postulating $\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{\xi})$ with internal variables $\boldsymbol{\xi}$, the Coleman–Noll argument gave:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^v, \qquad \boldsymbol{X} := -\frac{\partial \psi}{\partial \boldsymbol{\xi}}, \qquad \mathcal{D} = \boldsymbol{\sigma}^v : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \;\geq\; 0. \tag{2}$$

Strain-rate insensitivity implies $\boldsymbol{\sigma}^v = \mathbf{0}$. Thus, stress is purely energetic and dissipation reduces to

$$\mathcal{D} = \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \;\geq\; 0. \tag{3}$$

Formulating plasticity theory amounts to choosing $\boldsymbol{\xi}$ and its evolution.

Definition: Internal variables of plasticity

Plasticity consists of two internal variables, plastic strain and hardening variables,

$$\boldsymbol{\xi} = (\boldsymbol{\varepsilon}^p,\ \alpha). \tag{4}$$

Plastic strain $\boldsymbol{\varepsilon}^p$ is a symmetric tensor recording permanent macroscale deformation. Hardening variables α record microstructural changes that accompany plastic flow.

Definition: Elastic strain and the energy split

The free energy is postulated to depend on $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^p$ only through their difference, called the *elastic strain*. It also depends on the hardening variables separately,

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \alpha) = \psi^e(\boldsymbol{\varepsilon}^e) + \psi^h(\alpha), \qquad \boldsymbol{\varepsilon}^e := \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p. \tag{5}$$

The recoverable part of the free energy is the elastically stored portion $\psi^e(\boldsymbol{\varepsilon}^e)$.

Proposition: Conjugate forces of plasticity

Under the energy split, the conjugate forces of $\boldsymbol{\varepsilon}^p$ and α are:

$$\boldsymbol{\sigma} = \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e} = -\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^p}, \qquad A := -\frac{\partial \psi^h}{\partial \alpha} = -\frac{\partial \psi}{\partial \alpha}, \tag{6}$$

reducing the dissipation inequality to:

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + A \cdot \dot{\alpha} \;\geq\; 0. \tag{7}$$

The stress $\boldsymbol{\sigma}$ is macroscopic and measurable, while A is microscopic and not measurable.

Proof. Recall the definition of the force conjugate $\boldsymbol{X} = -\partial\psi/\partial\boldsymbol{\xi}$ from the previous lecture. Thus, the force conjugate to $\boldsymbol{\varepsilon}^p$ is $-\partial\psi/\partial\boldsymbol{\varepsilon}^p$, which under the energy split collapses to the stress,

$$-\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^p} \;=\; \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e} \;=\; \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} \;=\; \boldsymbol{\sigma}, \tag{8}$$

The first equality uses $\partial\boldsymbol{\varepsilon}^e/\partial\boldsymbol{\varepsilon}^p = -\boldsymbol{I}$, the second uses $\partial\boldsymbol{\varepsilon}^e/\partial\boldsymbol{\varepsilon} = \boldsymbol{I}$ with ψ^h independent of $\boldsymbol{\varepsilon}$, and the third is the stress relation with $\boldsymbol{\sigma}^v = \boldsymbol{0}$. The force conjugate to α is A by the same definition. Substituting $\boldsymbol{X} = (\boldsymbol{\sigma}, A)$ and $\dot{\boldsymbol{\xi}} = (\dot{\boldsymbol{\varepsilon}}^p, \dot{\alpha})$ into $\mathcal{D} = \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}}$ gives the inequality. □

Example. For linear elasticity, $\psi^e = \frac{1}{2}\,\boldsymbol{\varepsilon}^e : \mathbb{C} : \boldsymbol{\varepsilon}^e$. Therefore, $\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$.

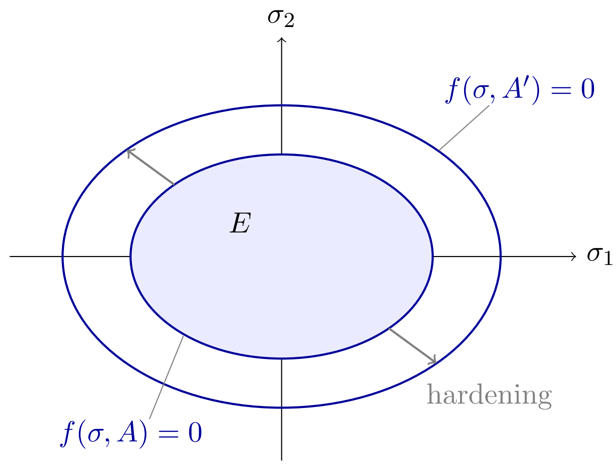
Definition: Yield function and elastic domain

Plasticity sets a threshold on forces. The yield function $f(\boldsymbol{\sigma}, A)$ delimits the elastic domain,

$$E = \{(\boldsymbol{\sigma}, A) \;\; f(\boldsymbol{\sigma}, A) \leq 0\}, \tag{9}$$

States with $f<0$ are elastic, plastic flow occurs only on $f=0$, and $f>0$ is inadmissible.

Although f is a function on the joint space of $\boldsymbol{\sigma}$ and A, it is often visualized through its slices at fixed A, drawn in stress space. Hardening is the motion of the slice as A evolves.



The previous lecture tied fluxes and forces through a dissipation potential, $\boldsymbol{X} = \partial\varphi/\partial\dot{\boldsymbol{\xi}}$. For plasticity, the flux is the internal-variable rate $\dot{\boldsymbol{\xi}} = (\dot{\varepsilon}^p, \dot{\alpha})$ and the force is $\boldsymbol{X} = (\boldsymbol{\sigma}, A)$. Therefore, the natural question is whether there exists some potential $\varphi(\dot{\boldsymbol{\xi}})$ that can reproduce a rate-independent response. The answer is *yes*, but such a potential cannot be differentiable at zero.

Definition: Positive homogeneity

A function $h(x)$ is positively homogeneous of degree n if it scales as follows:

$$h(c\,x) = c^n\,h(x) \qquad \forall c > 0. \tag{10}$$

Proposition: Rate independence forbids a smooth flux potential

Let $\varphi(\dot{\boldsymbol{\xi}})$ be a dissipation potential, convex with $\varphi \geq 0$ and $\varphi(\mathbf{0}) = 0$. Denote the conjugate force $\boldsymbol{X} = \partial\varphi/\partial\dot{\boldsymbol{\xi}}$. Rate-independent plasticity implies force is unchanged under the scaling

$$\boldsymbol{X}(c\,\dot{\boldsymbol{\xi}}) = \boldsymbol{X}(\dot{\boldsymbol{\xi}}) \qquad \forall c > 0,\ \dot{\boldsymbol{\xi}} \neq \mathbf{0}, \tag{11}$$

meaning $\boldsymbol{X}$ is positively homogeneous of degree zero. Then, it must hold that φ is positively homogeneous of degree one. Moreover, if φ is differentiable at $\dot{\boldsymbol{\xi}} = \mathbf{0}$, then $\varphi \equiv 0$. In other words, a nontrivial rate-independent potential cannot be differentiable at zero.

Proof. Fix $\dot{\boldsymbol{\xi}} \neq \mathbf{0}$ and set $h(c) := \varphi(c\,\dot{\boldsymbol{\xi}})$ for $c \geq 0$. Differentiating in c and using degree-zero force,

$$h'(c) = \left.\frac{\partial\varphi}{\partial\dot{\boldsymbol{\xi}}}\right|_{c\dot{\boldsymbol{\xi}}} \cdot \dot{\boldsymbol{\xi}} = \boldsymbol{X}(c\,\dot{\boldsymbol{\xi}}) \cdot \dot{\boldsymbol{\xi}} = \boldsymbol{X}(\dot{\boldsymbol{\xi}}) \cdot \dot{\boldsymbol{\xi}}, \tag{12}$$

which is a constant. Integrating from $c = 0$ with $h(0) = \varphi(\mathbf{0}) = 0$ gives $h(c) = c\,\boldsymbol{X}(\dot{\boldsymbol{\xi}})\cdot\dot{\boldsymbol{\xi}}$. Evaluating at $c = 1$ identifies $\varphi(\dot{\boldsymbol{\xi}}) = \boldsymbol{X}(\dot{\boldsymbol{\xi}}) \cdot \dot{\boldsymbol{\xi}}$. Thus, we prove φ is degree-one homogeneous by,

$$\varphi(c\dot{\boldsymbol{\xi}}) = h(c) = c\,\varphi(\dot{\boldsymbol{\xi}}), \tag{13}$$

Next, suppose φ is differentiable at the origin with gradient $\boldsymbol{X}_0$. This means the first-order Taylor expansion holds with a remainder vanishing faster than its argument (Landau little-o notation),

$$\varphi(\boldsymbol{\eta}) = \boldsymbol{X}_0 \cdot \boldsymbol{\eta} + o(|\boldsymbol{\eta}|), \qquad \frac{o(|\boldsymbol{\eta}|)}{|\boldsymbol{\eta}|} \to 0 \ \text{ as } \boldsymbol{\eta} \to \boldsymbol{0}. \tag{14}$$

Substituting $\boldsymbol{\eta} = c\,\dot{\boldsymbol{\xi}}$ and dividing by c, then using degree-one homogeneity, we obtain:

$$\varphi(\dot{\boldsymbol{\xi}}) = \frac{\varphi(c\,\dot{\boldsymbol{\xi}})}{c} = \boldsymbol{X}_0 \cdot \dot{\boldsymbol{\xi}} + \frac{o(c\,|\dot{\boldsymbol{\xi}}|)}{c}. \tag{15}$$

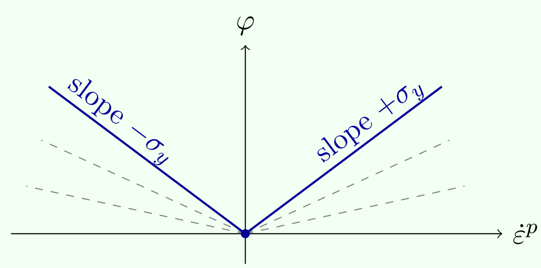
The left side does not depend on c, and the remainder tends to zero as $c \to 0^+$. Hence, $\varphi(\dot{\boldsymbol{\xi}}) = \boldsymbol{X}_0 \cdot \dot{\boldsymbol{\xi}}$ holds for every $\dot{\boldsymbol{\xi}}$, which means φ is linear. But linearity and nonnegativity are incompatible unless φ vanishes (seen by flipping the sign of $\dot{\boldsymbol{\xi}}$). Thus, a non-trivial φ is not differentiable at zero. □

Example: The one-dimensional bar

Consider a bar with one internal variable, $\dot{\boldsymbol{\xi}} = \dot{\varepsilon}^p$ and $\boldsymbol{X} = \sigma$. Degree-one homogeneity fully determines the shape of the potential. For $\dot{\varepsilon}^p > 0$, taking $c = \dot{\varepsilon}^p$ in the definition gives $\varphi(\dot{\varepsilon}^p) = \dot{\varepsilon}^p\,\varphi(1)$. For $\dot{\varepsilon}^p < 0$, the definition (acting only through positive factors) gives $\varphi(\dot{\varepsilon}^p) = |\dot{\varepsilon}^p|\,\varphi(-1)$. Nonnegativity requires $\varphi(1) \geq 0$ and $\varphi(-1) \geq 0$. If we assume equal thresholds in tension and compression, then $\varphi(1) = \varphi(-1) =: \sigma_y$ and we obtain:

$$\varphi(\dot{\varepsilon}^p) = \sigma_y\,|\dot{\varepsilon}^p|, \tag{16}$$

a cone with its corner at zero.



The above potential displays three key features: (1) During flow the force is pinned,

$$\sigma = X = \frac{d\varphi}{d\dot{\varepsilon}^p} = \sigma_y\ \mathrm{sign}(\dot{\varepsilon}^p) \qquad (\dot{\varepsilon}^p \neq 0), \tag{17}$$

independent of speed $|\dot{\varepsilon}^p|$. (2) At zero flux the derivative does not exist, with one-sided slopes $-\sigma_y$ and $+\sigma_y$. Thus, φ leaves the force at rest undetermined within $[-\sigma_y, \sigma_y]$, which is the elastic domain. (3) If we attempt to invert $\sigma = \sigma_y\ \mathrm{sign}(\dot{\varepsilon}^p)$ to express flux as a function of force, given $\sigma = \pm\sigma_y$ determines the sign of the flux $\dot{\varepsilon}^p$ but never its magnitude.

The general theory of plasticity postulates exactly the above structure derived in 1D: (1) The pinning of force during flow becomes the KKT conditions below, with flow occurring only on the boundary of the admissible set. (2) The interval of sustainable forces becomes the elastic domain E, already introduced. The yield function f is its smooth description, with membership tested by the sign of f and the boundary's outward normal given by its gradient. (3) The undetermined magnitude becomes a flow rule that prescribes only the direction of flux, through the gradient of a scalar function g of forces. The magnitude is carried by a new unknown, the plastic multiplier.

Definition: Flow rule and KKT conditions

The internal variables evolve along the gradient of a plastic potential $g(\boldsymbol{\sigma}, A)$,

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\frac{\partial g}{\partial \boldsymbol{\sigma}}, \qquad \dot{\alpha} = \dot{\gamma}\,\frac{\partial g}{\partial A}, \tag{18}$$

with the plastic multiplier $\dot{\gamma}$ subject to the Karush–Kuhn–Tucker (KKT) conditions:

$$\dot{\gamma} \geq 0, \qquad f \leq 0, \qquad \dot{\gamma}\, f = 0. \tag{19}$$

The flow rule is associative if $g = f$, and non-associative otherwise. Inside E, $\dot{\gamma} = 0$ holds and the response is elastic. During plastic flow, $\dot{\gamma} > 0$ and the state is on the yield surface.

Proposition: Associative flow satisfies dissipation inequality

Let flow be associative, $g = f$. And let f be convex in $(\boldsymbol{\sigma}, A)$ with the rest state admissible, $f(\mathbf{0}, 0) \leq 0$. Then, the KKT conditions imply:

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + A \cdot \dot{\alpha} \;\geq\; 0 \tag{20}$$

Proof. If $\dot{\gamma} = 0$ then $\mathcal{D} = 0$. If $\dot{\gamma} > 0$, $\boldsymbol{X} = (\boldsymbol{\sigma}, A)$ satisfies $f(\boldsymbol{X}) = 0$, and the flow rule gives:

$$\mathcal{D} = \dot{\gamma}\;\boldsymbol{X} \cdot \frac{\partial f}{\partial \boldsymbol{X}}. \tag{21}$$

From the convexity of f, the following holds:

$$0 \geq f(\mathbf{0}) \;\geq\; f(\boldsymbol{X}) + \frac{\partial f}{\partial \boldsymbol{X}} \cdot (\mathbf{0} - \boldsymbol{X}) \;=\; -\boldsymbol{X} \cdot \frac{\partial f}{\partial \boldsymbol{X}}, \tag{22}$$

Hence, $\boldsymbol{X} \cdot \partial f / \partial \boldsymbol{X} \geq 0$, and $\mathcal{D} \geq 0$ follows. $\square$

Remark: Non-associative flow and dissipation inequality

For non-associative flow, $g \neq f$, the inequality $\mathcal{D} \geq 0$ is not automatic and must be checked for each model. Even granting g convexity, the argument above breaks in two places. Repeating it with g gives $\boldsymbol{X} \cdot \partial g/\partial \boldsymbol{X} \geq g(\boldsymbol{X}) - g(\mathbf{0})$, but during yield $g(\boldsymbol{X})$ is not necessarily zero, since the state lies on the surface of f, not g. And $g(\mathbf{0})$ is not necessarily negative, since only f was required to contain the rest state. Both were needed to conclude $\mathcal{D} \geq 0$.

Proposition: Rate independence follows from flow rule and KKT

The elastoplastic model implied by the flow rule and the KKT conditions is rate independent.

Proof. Let $\boldsymbol{\varepsilon}(t)$, $\boldsymbol{\varepsilon}^p(t)$, $\alpha(t)$, $\gamma(t)$ solve the constitutive equations. Traversing the same strain path faster implies setting $\hat{\boldsymbol{\varepsilon}}(t) := \boldsymbol{\varepsilon}(ct)$ with $c > 0$, and $\hat{\boldsymbol{\varepsilon}}^p(t) := \boldsymbol{\varepsilon}^p(ct)$, $\hat{\alpha}(t) := \alpha(ct)$, $\hat{\gamma}(t) := \gamma(ct)$. Each hatted rate picks up the factor c, and flow rules hold with multiplier $\dot{\hat{\gamma}} = c\,\dot{\gamma}$:

$$\dot{\hat{\boldsymbol{\varepsilon}}}^p(t) = c\,\dot{\boldsymbol{\varepsilon}}^p(ct) = c\,\dot{\gamma}(ct)\,\frac{\partial g}{\partial \boldsymbol{\sigma}}\bigg|_{ct} = \dot{\hat{\gamma}}(t)\,\frac{\partial g}{\partial \boldsymbol{\sigma}}\bigg|_{ct}, \tag{23}$$

The $\boldsymbol{\sigma} = \partial\psi^e/\partial\boldsymbol{\varepsilon}^e$, $f \leq 0$, and $\dot{\gamma} f = 0$ are unchanged as they contain no rates. Thus, hatted fields solve the same equations for input $\hat{\boldsymbol{\varepsilon}}$, and by uniqueness give the same response. Hence, $\hat{\boldsymbol{\sigma}}(t) = \boldsymbol{\sigma}(ct)$ follows from $\hat{\boldsymbol{\varepsilon}}^e(t) = \boldsymbol{\varepsilon}^e(ct)$, so the sped-up test traces the same stress–strain curve. □

2. Maximum plastic dissipation

This section shows flow rule and KKT conditions are optimality conditions of an optimization. For a given flux, the principle of maximum plastic dissipation asserts that, among all force states permitted by the yield condition, the realized state is the one that dissipates the most.

Theorem: Maximum plastic dissipation

Suppose the yield function f is smooth and convex, with $\partial f/\partial \boldsymbol{X} \neq \boldsymbol{0}$ wherever $f = 0$. Let a flux $\dot{\boldsymbol{\xi}} = (\dot{\boldsymbol{\varepsilon}}^p, \dot{\alpha})$ be given and let $\boldsymbol{X} = (\boldsymbol{\sigma}, A)$ be an admissible state, $f(\boldsymbol{X}) \leq 0$. Then the following two conditions on the pair $(\boldsymbol{X}, \dot{\boldsymbol{\xi}})$ are equivalent.

(i) The state maximizes the dissipation over E:

$$\boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \;\geq\; \boldsymbol{X}^* \cdot \dot{\boldsymbol{\xi}} \qquad \forall \boldsymbol{X}^* \text{ with } f(\boldsymbol{X}^*) \leq 0. \tag{24}$$

(ii) There exists a $\dot{\gamma} \geq 0$ such that *associative* flow holds:

$$\dot{\boldsymbol{\xi}} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{X}}\bigg|_{\boldsymbol{X}} \qquad \text{and} \qquad \dot{\gamma}\,f(\boldsymbol{X}) = 0. \tag{25}$$

Proof. (ii) ⇒ (i). If $\dot{\gamma} = 0$ then $\dot{\boldsymbol{\xi}} = \boldsymbol{0}$ and both sides of the inequality vanish. If $\dot{\gamma} > 0$ then $f(\boldsymbol{X}) = 0$ by complementarity, and for any admissible $\boldsymbol{X}^*$, convexity gives:

$$f(\boldsymbol{X}^*) \;\geq\; f(\boldsymbol{X}) + \frac{\partial f}{\partial \boldsymbol{X}} \cdot (\boldsymbol{X}^* - \boldsymbol{X}) \;=\; \frac{\partial f}{\partial \boldsymbol{X}} \cdot (\boldsymbol{X}^* - \boldsymbol{X}). \tag{26}$$

Multiplying by $\dot{\gamma} \geq 0$ and using the flow rule,

$$\dot{\gamma}\,f(\boldsymbol{X}^*) \;\geq\; \dot{\boldsymbol{\xi}} \cdot (\boldsymbol{X}^* - \boldsymbol{X}), \tag{27}$$

and the left side is nonpositive since $f(\boldsymbol{X}^*) \leq 0$. Hence $\boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \geq \boldsymbol{X}^* \cdot \dot{\boldsymbol{\xi}}$.

(i) ⇒ (ii). If $\dot{\boldsymbol{\xi}} = \boldsymbol{0}$, take $\dot{\gamma} = 0$ and both conditions hold. If $\dot{\boldsymbol{\xi}} \neq \boldsymbol{0}$, we must show three things:

First, we show $\boldsymbol{X}$ lies on the yield surface. The quantity being maximized, $\boldsymbol{X}^* \mapsto \boldsymbol{X}^* \cdot \dot{\boldsymbol{\xi}}$, increases along the direction $\dot{\boldsymbol{\xi}}$. So if $\boldsymbol{X}$ were an interior point of E, small steps in every direction would remain admissible and result in an admissible state with larger dissipation, contradicting maximality. Hence $f(\boldsymbol{X}) = 0$ holds, and the complementarity condition $\dot{\gamma} f(\boldsymbol{X}) = 0$ is satisfied for any multiplier.

Second, we show $\dot{\boldsymbol{\xi}}$ has no component tangent to the yield surface. Write $\boldsymbol{n} := \partial f/\partial \boldsymbol{X}|_{\boldsymbol{X}} \neq \boldsymbol{0}$ for the outward normal and split the flux into a normal part and a tangential remainder:

$$\dot{\boldsymbol{\xi}} = \dot{\gamma}\,\boldsymbol{n} + \boldsymbol{w}, \qquad \dot{\gamma} := \frac{\dot{\boldsymbol{\xi}} \cdot \boldsymbol{n}}{|\boldsymbol{n}|^2}, \qquad \boldsymbol{w} \cdot \boldsymbol{n} = 0, \tag{28}$$

The claim to prove is $\boldsymbol{w} = \boldsymbol{0}$ together with $\dot{\gamma} \geq 0$. The argument uses the fact that all admissible perturbations of $\boldsymbol{X}$ must not increase the dissipation. Let $\boldsymbol{d}$ with $\boldsymbol{d} \cdot \boldsymbol{n} < 0$ be any direction that points into E. By Taylor expansion of f at $\boldsymbol{X}$ along $\boldsymbol{d}$, and using $f(\boldsymbol{X}) = 0$, we get:

$$f(\boldsymbol{X} + t\,\boldsymbol{d}) = f(\boldsymbol{X}) + t\,\frac{\partial f}{\partial \boldsymbol{X}} \cdot \boldsymbol{d} + o(t) = t\,\boldsymbol{d} \cdot \boldsymbol{n} + o(t), \tag{29}$$

Note $f(\boldsymbol{X} + t\,\boldsymbol{d}) < 0$ because the linear term is negative and dominates for small $t > 0$. Thus, the perturbed state $\boldsymbol{X} + t\,\boldsymbol{d}$ is admissible and maximum principle gives $(\boldsymbol{X} + t\,\boldsymbol{d}) \cdot \dot{\boldsymbol{\xi}} \leq \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}}$, which yields $\boldsymbol{d} \cdot \dot{\boldsymbol{\xi}} \leq 0$. The tangential part $\boldsymbol{w}$ itself is not such a direction, $\boldsymbol{w} \cdot \boldsymbol{n} = 0$, but tilting it slightly inward is. Testing $\boldsymbol{d} = \boldsymbol{w} - \epsilon\,\boldsymbol{n}$ with $\epsilon > 0$, for which $\boldsymbol{d} \cdot \boldsymbol{n} = -\epsilon\,|\boldsymbol{n}|^2 < 0$, gives:

$$0 \;\geq\; (\boldsymbol{w} - \epsilon\,\boldsymbol{n}) \cdot \dot{\boldsymbol{\xi}} \;=\; |\boldsymbol{w}|^2 - \epsilon\,\dot{\gamma}\,|\boldsymbol{n}|^2 \qquad \forall \epsilon > 0, \tag{30}$$

Letting $\epsilon \to 0^+$ forces $|\boldsymbol{w}|^2 \leq 0$, implying $\boldsymbol{w} = \boldsymbol{0}$.

Third, we show $\dot{\gamma} > 0$. Perturbing with the direction $\boldsymbol{d} = -\boldsymbol{n}$ gives $-\boldsymbol{n} \cdot \dot{\boldsymbol{\xi}} = -\dot{\gamma}\,|\boldsymbol{n}|^2 \leq 0$, which proves $\dot{\gamma} \geq 0$. However, because $\dot{\boldsymbol{\xi}} = \dot{\gamma}\,\boldsymbol{n} \neq \boldsymbol{0}$, we must have $\dot{\gamma} > 0$. □

If maximum principle holds, associativity is a consequence not a modeling choice. Condition (ii) states the flux points along the outward normal of ∂E. Thus, associativity is also called *normality*.

Corollary: Consequences of maximum principle

Maximum principle implies the dissipation inequality, provided $f(\boldsymbol{0}) \leq 0$ which is sensible:

$$\mathcal{D} = \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \;\geq\; 0. \tag{31}$$

Proof. Choose the comparison state $\boldsymbol{X}^* = \boldsymbol{0}$ in condition (i) of the maximum principle theorem, admissible by the hypothesis. The maximum property gives $\mathcal{D} = \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \geq \boldsymbol{0} \cdot \dot{\boldsymbol{\xi}} = 0$. □

Remark: Recovering the lost potential

What became of the dissipation potential? The maximum principle provides a way of answering that. Specifically, define the following function:

$$\varphi(\dot{\boldsymbol{\xi}}) \;:=\; \max_{f(\boldsymbol{X}^*) \leq 0} \;\boldsymbol{X}^* \cdot \dot{\boldsymbol{\xi}}. \tag{32}$$

which has every property a dissipation potential has. It is nonnegative, because $\boldsymbol{X}^* = \boldsymbol{0}$ is admissible and contributes $\boldsymbol{0} \cdot \dot{\boldsymbol{\xi}} = 0$, so the maximum is at least zero. It vanishes at zero flux. It is convex, being a maximum of linear functions of $\dot{\boldsymbol{\xi}}$. It is positively homogeneous of degree one, since scaling $\dot{\boldsymbol{\xi}}$ by $c > 0$ scales every argument $\boldsymbol{X}^* \cdot \dot{\boldsymbol{\xi}}$ by c. Thus, it is not smooth at zero flux. When flow is associative, the above theorem ensures the actual dissipation *attains* the maximum, $\varphi(\dot{\boldsymbol{\xi}}) = \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}}$. Non-associative flow breaks this equality (see Appendix A).

3. Consistency condition and plastic multiplier

The KKT conditions decide when plastic flow occurs but not how much, namely $\dot{\gamma}$. During flow the state stays on the yield surface, $f = 0$, and differentiating that yields $\dot{\gamma}$ in closed form.

Proposition: The plastic multiplier

During plastic flow, the multiplier is obtained from:

$$\dot{\gamma} \;=\; \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : C : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : C : \dfrac{\partial g}{\partial \boldsymbol{\sigma}} \;+\; H}, \qquad H := \frac{\partial f}{\partial A} \cdot \frac{\partial^2 \psi^h}{\partial \alpha^2} \cdot \frac{\partial g}{\partial A} \tag{33}$$

where $C := \partial^2 \psi^e / \partial \boldsymbol{\varepsilon}^e \partial \boldsymbol{\varepsilon}^e$ is the elastic stiffness and H is the hardening modulus.

Proof. During flow $\dot{\gamma} > 0$, so complementarity pins $f = 0$ at every instant of the interval. Taking the derivative gives the consistency condition:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial A} \cdot \dot{A} = 0. \tag{34}$$

The stress rate follows from $\boldsymbol{\sigma} = \partial \psi^e / \partial \boldsymbol{\varepsilon}^e$ and the flow rule:

$$\dot{\boldsymbol{\sigma}} = C : \dot{\boldsymbol{\varepsilon}}^e = C : \Big(\dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \frac{\partial g}{\partial \boldsymbol{\sigma}} \Big), \tag{35}$$

and the hardening force rate from $A = -\partial \psi^h / \partial \alpha$ and the flow rule:

$$\dot{A} = -\frac{\partial^2 \psi^h}{\partial \alpha^2} \cdot \dot{\alpha} = -\dot{\gamma} \frac{\partial^2 \psi^h}{\partial \alpha^2} \cdot \frac{\partial g}{\partial A}. \tag{36}$$

Substituting both into the consistency condition gives:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : C : \dot{\boldsymbol{\varepsilon}} \;-\; \dot{\gamma} \left[\frac{\partial f}{\partial \boldsymbol{\sigma}} : C : \frac{\partial g}{\partial \boldsymbol{\sigma}} + H \right] = 0. \tag{37}$$

Solving for $\dot{\gamma}$ proves the formula. □

The formula has the following interpretation. If $\dot{\gamma} = 0$, the yield function changes at trial rate:

$$\dot{f}^{\,tr} := \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : C : \dot{\boldsymbol{\varepsilon}}, \tag{38}$$

which is the numerator. At a force state on the yield surface, if $\dot{f}^{\,tr} \leq 0$ the change is admissible, the step is elastic, $\dot{\gamma} = 0$, and the formula is not invoked. If $\dot{f}^{\,tr} > 0$ the change would violate $f \leq 0$, plastic flow activates, and the formula returns $\dot{\gamma} > 0$. Since numerator is positive, denominator must be positive too. For associative flow, the first term in the denominator is positive provided C is positive definite and $\partial f / \partial \boldsymbol{\sigma} \neq \mathbf{0}$. Hardening H must be such that the sum stays positive.

4. J_2 plasticity

With the general theory complete, we apply it to metal plasticity. Experiments show that hydrostatic pressure does not make metals yield, so the yield function depends on the stress deviator alone. The resulting model is due to Richard von Mises 1913, and is called J_2 plasticity.

Definition: Deviatoric stress invariants

From first lecture, stress splits into mean and deviatoric components:

$$\boldsymbol{\sigma} = \sigma_m \mathbf{1} + \boldsymbol{s}, \qquad \sigma_m := \tfrac{1}{3}\operatorname{tr}\boldsymbol{\sigma}, \qquad \operatorname{tr}\boldsymbol{s} = 0. \tag{39}$$

The magnitude of $\boldsymbol{s}$ is quantified by the second invariant and Frobenius norm:

$$J_2 := \tfrac{1}{2}(\boldsymbol{s} : \boldsymbol{s}), \qquad \|\boldsymbol{s}\| := \sqrt{\boldsymbol{s} : \boldsymbol{s}} = \sqrt{2J_2}, \tag{40}$$

The direction of $\boldsymbol{s}$ is captured by $\boldsymbol{Q} := \boldsymbol{s}/\|\boldsymbol{s}\|$, which satisfies $\|\boldsymbol{Q}\| = 1$ and $\operatorname{tr}\boldsymbol{Q} = 0$.

Example. For isotropic elasticity, $\sigma_m = K\,\varepsilon_v^e$ and $\boldsymbol{s} = 2\mu\,\boldsymbol{e}^e$, a form used repeatedly below.

Remark: Trace identities

The following useful trace identities hold:

$$\operatorname{tr}\boldsymbol{A} = \operatorname{tr}\boldsymbol{A}^\top, \qquad \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B}\boldsymbol{A}), \qquad \operatorname{tr}(\boldsymbol{A}^\top\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B}^\top\boldsymbol{A}) = \boldsymbol{A} : \boldsymbol{B}. \tag{41}$$

Remark: Where J_2 comes from

Every tensor $\boldsymbol{\sigma}$ has three principal invariants:

$$I_1 = \operatorname{tr}\boldsymbol{\sigma}, \quad I_2 = \tfrac{1}{2}\left[(\operatorname{tr}\boldsymbol{\sigma})^2 - \operatorname{tr}(\boldsymbol{\sigma}^2)\right], \quad I_3 = \det\boldsymbol{\sigma}. \tag{42}$$

which are the coefficients of its characteristic polynomial:

$$\det(\boldsymbol{\sigma} - \lambda\mathbf{1}) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 \tag{43}$$

The J notation applies these to $\boldsymbol{s}$, with a minus sign chosen to make the quadratic invariant positive, $J_1 = \operatorname{tr}\boldsymbol{s} = 0$, $J_2 := -I_2(\boldsymbol{s}) = \operatorname{tr}(\boldsymbol{s}^2)/2 = (\boldsymbol{s} : \boldsymbol{s})/2$, $J_3 := \det\boldsymbol{s}$. Two useful equivalent forms are $J_2 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2)$ and $J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$. The latter shows J_2 measures differences in principal stresses and ignores pressure.

Definition: The von Mises model

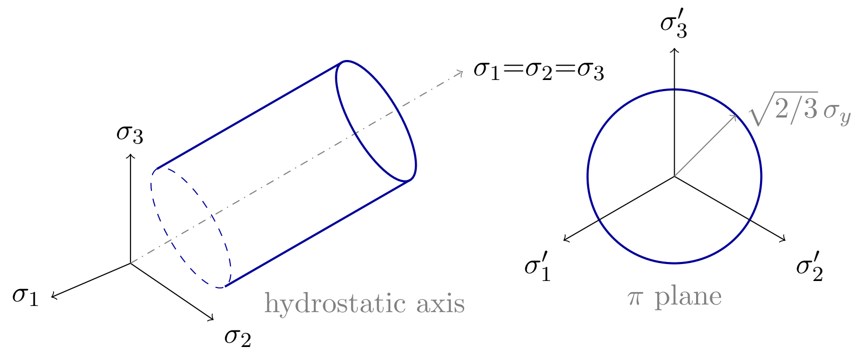
The von Mises model is associative, $g = f$, and the yield function defined by:

$$f(\boldsymbol{\sigma}, A) = \sqrt{3J_2} - \underbrace{(\sigma_{y0} - A)}_{\sigma_y(\alpha)}, \qquad \psi^h = \psi^h(\alpha), \qquad A = -\frac{d\psi^h}{d\alpha}, \tag{44}$$

with σ_{y0} the initial yield stress, and $\psi^h(\alpha)$ (arbitrary) and A introduced earlier.

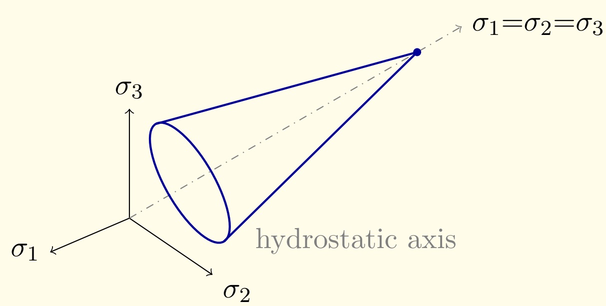
Example. Linear hardening takes $\psi^h = \frac{1}{2}H\alpha^2$ with constant $H > 0$, giving $\sigma_y = \sigma_{y0} + H\alpha$.

At fixed A the yield surface is a cylinder about the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$ in principal-stress space. The reason is that a stress point $(\sigma_1, \sigma_2, \sigma_3)$ splits as $\sigma_m(1, 1, 1) + (s_1, s_2, s_3)$, and since the deviator's principal values sum to zero, the second vector is perpendicular to $(1, 1, 1)$. The distance from the point to the hydrostatic axis is therefore the length of that perpendicular part, which is $\|\boldsymbol{s}\|$. Note the Frobenius norm is invariant under coordinate transform, and equal to Euclidean length of principal values. Hence, the yield surface $f = 0$ defines a cylinder of radius $\|\boldsymbol{s}\| = \sqrt{2J_2} = \sqrt{2/3}\,\sigma_y$. Hardening dilates it. The plane normal to the axis is called the π plane.



Remark: Pressure-dependent yield

The von Mises cylinder never closes on the hydrostatic axis, so pressure cannot cause yield. Soils, rocks, and other frictional materials yield under pressure and $f = 0$ defines a cone or capped cone about the hydrostatic axis. Such models often require non-associative flow, a departure from the potential framework above and beyond the scope of this lecture.



Example: Uniaxial tension and pure shear

For uniaxial tension, $\boldsymbol{\sigma} = \sigma\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ and the following hold:

$$\boldsymbol{s} = \operatorname{diag}\left(\tfrac{2\sigma}{3}, -\tfrac{\sigma}{3}, -\tfrac{\sigma}{3}\right), \qquad J_2 = \tfrac{\sigma^2}{3} \implies |\sigma| = \sqrt{3J_2}, \tag{45}$$

so yielding in tension occurs at $\sigma = \sigma_y$ as intended.
For pure shear, $\boldsymbol{\sigma} = \tau\, (\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)$ and the following hold:

$$\boldsymbol{s} = \boldsymbol{\sigma}, \qquad J_2 = \tau^2 \implies |\tau| = \sqrt{J_2}, \tag{46}$$

so yielding occurs at $\tau_y = \sigma_y/\sqrt{3} \approx 0.577\, \sigma_y$. The above shear-yield stress is a prediction of the model, not an input, and experiments on ductile metals confirm its validity.

Proposition: Flow direction and plastic incompressibility

For the von Mises model, the gradient of the yield function is:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \sqrt{\tfrac{3}{2}}\, \boldsymbol{Q}, \tag{47}$$

resulting in the following flow rules:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \sqrt{\tfrac{3}{2}}\, \boldsymbol{Q}, \qquad \operatorname{tr} \dot{\boldsymbol{\varepsilon}}^p = 0, \qquad \dot{\alpha} = \dot{\gamma}. \tag{48}$$

Second equation implies plastic flow is incompressible. The accumulated plastic strain is:

$$\bar{\varepsilon}^p := \int \sqrt{\tfrac{2}{3}}\, \|\dot{\boldsymbol{\varepsilon}}^p\| \ \mathrm{d}t, \qquad \dot{\bar{\varepsilon}}^p = \dot{\gamma}. \tag{49}$$

Proof. First, $\partial J_2/\partial \boldsymbol{\sigma} = \boldsymbol{s}$. In components, $J_2 = \frac{1}{2} s_{nm} s_{nm}$ with $s_{nm} = \sigma_{nm} - \frac{1}{3}\sigma_{kk}\delta_{nm}$. Hence:

$$\frac{\partial J_2}{\partial \sigma_{ij}} = s_{nm} \frac{\partial s_{nm}}{\partial \sigma_{ij}} = s_{nm}\Big(\delta_{ni}\delta_{mj} - \tfrac{1}{3}\,\delta_{ij}\,\delta_{nm}\Big) = s_{ij} - \tfrac{1}{3}\,\delta_{ij} \operatorname{tr} \boldsymbol{s} = s_{ij}. \tag{50}$$

Second, by chain rule and using $f = \sqrt{3J_2} - (\sigma_{y0} - A) = \sqrt{3/2}\, \|\boldsymbol{s}\| - (\sigma_{y0} - A)$, we get:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3}{2\sqrt{3J_2}} \frac{\partial J_2}{\partial \boldsymbol{\sigma}} = \frac{3\, \boldsymbol{s}}{2\sqrt{3/2}\, \|\boldsymbol{s}\|} = \sqrt{\tfrac{3}{2}}\, \boldsymbol{Q}. \tag{51}$$

Third, from $\partial f/\partial A = 1$, we get $\dot{\alpha} = \dot{\gamma}$. Since $\boldsymbol{Q}$ is deviatoric, $\operatorname{tr} \dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\sqrt{3/2}\ \operatorname{tr} \boldsymbol{Q} = 0$. Finally, $\|\dot{\boldsymbol{\varepsilon}}^p\| = \dot{\gamma}\sqrt{3/2}\, \|\boldsymbol{Q}\| = \dot{\gamma}\sqrt{3/2}$, so $\sqrt{2/3}\, \|\dot{\boldsymbol{\varepsilon}}^p\| = \dot{\gamma}$, and integrating from an intact state, $\bar{\varepsilon}^p = \gamma$. $\square$

Remark: The $\sqrt{2/3}$ factor in $\bar{\varepsilon}^p$

The $\sqrt{2/3}$ factor makes $\bar{\varepsilon}^p$ equal the axial plastic strain in uniaxial tension. There, transverse symmetry gives $\dot{\boldsymbol{\varepsilon}}^p = \operatorname{diag}(\dot{\varepsilon}^p_a,\ \dot{\varepsilon}^p_t,\ \dot{\varepsilon}^p_t)$, and incompressibility sets $\dot{\varepsilon}^p_t = -\dot{\varepsilon}^p_a/2$. Hence:

$$\|\dot{\boldsymbol{\varepsilon}}^p\| = |\dot{\varepsilon}^p_a|\sqrt{1+\tfrac{1}{4}+\tfrac{1}{4}} = |\dot{\varepsilon}^p_a|\sqrt{\tfrac{3}{2}} \quad\Longrightarrow\quad \dot{\varepsilon}^p_a = \sqrt{\tfrac{2}{3}}\,\|\dot{\boldsymbol{\varepsilon}}^p\| = \dot{\bar{\varepsilon}}^p, \tag{52}$$

where $\dot{\varepsilon}^p_a > 0$ under tension. Integrating makes $\bar{\varepsilon}^p$ the accumulated axial plastic strain.

Proposition: Plastic multiplier for J_2 model and isotropic elasticity

Assuming isotropic elasticity, the plastic multiplier for the J_2 model is:

$$\dot{\gamma} = \frac{\sqrt{6}\,\mu\,(\boldsymbol{Q}:\dot{\boldsymbol{e}})}{3\mu + H}, \qquad H = \frac{d\sigma_y}{d\bar{\varepsilon}^p} = \frac{d^2\psi^h}{d\alpha^2}, \tag{53}$$

with $\dot{\boldsymbol{e}}$ the deviatoric strain rate.

Proof. For the numerator, isotropy means $C:\dot{\boldsymbol{\varepsilon}} = K\,\dot{\varepsilon}_v\,\mathbf{1} + 2\mu\,\dot{\boldsymbol{e}}$, so

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : C : \dot{\boldsymbol{\varepsilon}} = \sqrt{\tfrac{3}{2}}\,\boldsymbol{Q} : \big(K\dot{\varepsilon}_v\mathbf{1} + 2\mu\,\dot{\boldsymbol{e}}\big) = \sqrt{6}\,\mu\,(\boldsymbol{Q}:\dot{\boldsymbol{e}}), \tag{54}$$

because $\boldsymbol{Q}:\mathbf{1} = \operatorname{tr}\boldsymbol{Q} = 0$. For the denominator, we first note the following identity:

$$C:\boldsymbol{Q} = K\,(\operatorname{tr}\boldsymbol{Q})\,\mathbf{1} + 2\mu\ \operatorname{dev}\boldsymbol{Q} = 2\mu\,\boldsymbol{Q}, \tag{55}$$

then use it to compute:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : C : \frac{\partial f}{\partial \boldsymbol{\sigma}} = \tfrac{3}{2}\,\boldsymbol{Q}:(C:\boldsymbol{Q}) = \tfrac{3}{2}\,\boldsymbol{Q}:(2\mu\,\boldsymbol{Q}) = 3\mu\,(\boldsymbol{Q}:\boldsymbol{Q}) = 3\mu. \tag{56}$$

As for the hardening modulus, $H = (\partial f/\partial A)\,(d^2\psi^h/d\alpha^2)\,(\partial f/\partial A) = d^2\psi^h/d\alpha^2$ since $\partial f/\partial A = 1$. Moreover, $\sigma_y(\alpha) = \sigma_{y0} - A = \sigma_{y0} + d\psi^h/d\alpha$ with $\alpha = \bar{\varepsilon}^p$, so differentiating yields $d\sigma_y/d\bar{\varepsilon}^p = H$. ☐

Example. For perfect plasticity, $H = 0$ and the multiplier reduces to $\dot{\gamma} = \sqrt{2/3}\,(\boldsymbol{Q}:\dot{\boldsymbol{e}})$.

Remark: Strain and work hardening

Hardening laws are stated in one of two arguments. Strain hardening, used above, takes $\sigma_y = \sigma_y(\bar{\varepsilon}^p)$. Work hardening takes $\sigma_y = \sigma_y(W^p)$ with $\dot{W}^p := \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^p$ as the plastic power. For J_2 flow the two coincide up to a reparametrization:

$$\dot{W}^p = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^p = (\sigma_m\mathbf{1} + \boldsymbol{s}):\dot{\boldsymbol{\varepsilon}}^p = \boldsymbol{s}:\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\sqrt{\tfrac{3}{2}}\,(\boldsymbol{s}:\boldsymbol{Q}) = \dot{\gamma}\sqrt{\tfrac{3}{2}}\,\|\boldsymbol{s}\| = \sigma_y\,\dot{\bar{\varepsilon}}^p, \tag{57}$$

A. Nonsmooth dissipation potentials

The main text showed rate-independent dissipation potentials are not differentiable at zero flux, and postulated flow rules to sidestep them. We develop the calculus of such nonsmooth potentials.

Definition: Subdifferential

For a convex φ, a slope $\boldsymbol{X}_0$ *supports* φ at $\dot{\boldsymbol{\xi}}_0$ if the graph lies above the line:

$$\varphi(\dot{\boldsymbol{\xi}}) \;\geq\; \varphi(\dot{\boldsymbol{\xi}}_0) + \boldsymbol{X}_0 \cdot (\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\xi}}_0) \qquad \forall \dot{\boldsymbol{\xi}}. \tag{58}$$

The set of supporting slopes is the subdifferential $\partial\varphi(\dot{\boldsymbol{\xi}}_0)$. If φ is differentiable at a point, the set contains only the gradient. The generalized force-flux relation is $\boldsymbol{X} \in \partial\varphi(\dot{\boldsymbol{\xi}})$.

For the 1D cone $\varphi = \sigma_y|\dot{\varepsilon}^p|$, the subdifferential at origin is the interval $\partial\varphi(0) = [-\sigma_y, \sigma_y]$, which coincidentally is the elastic domain. The generalized force-flux relation then reproduces both features of the 1D potential in the main text: pinned force $\pm\sigma_y$ during flow and full interval at rest.

Definition: The dual of dissipation potential

The Legendre transform of φ encodes the same law as a function of force:

$$\varphi^*(\boldsymbol{X}) \;:=\; \sup_{\dot{\boldsymbol{\xi}}} \left[\, \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} - \varphi(\dot{\boldsymbol{\xi}}) \,\right]. \tag{59}$$

For smooth strictly convex φ the supremum sits where $\boldsymbol{X} = \partial\varphi/\partial\dot{\boldsymbol{\xi}}$, which if inverted gives the map $\boldsymbol{X} \mapsto \dot{\boldsymbol{\xi}}$. The supremum form needs no derivative, so it survives corners, and the flux is recovered by differentiating in force space $\dot{\boldsymbol{\xi}} = \partial\varphi^*/\partial\boldsymbol{X}$ where the transform is smooth.

Example. For quadratic potential $\varphi(\dot{\boldsymbol{\xi}}) = \frac{1}{2}\,\dot{\boldsymbol{\xi}} \cdot \boldsymbol{M}^{-1}\dot{\boldsymbol{\xi}}$, maximizing gives $\dot{\boldsymbol{\xi}} = \boldsymbol{M}\boldsymbol{X}$ and substituting into dual gives $\varphi^*(\boldsymbol{X}) = \frac{1}{2}\,\boldsymbol{X} \cdot \boldsymbol{M}\boldsymbol{X}$. The gradient yields $\dot{\boldsymbol{\xi}} = \boldsymbol{M}\boldsymbol{X}$ again. One law, two encodings.

Example. For the cone, with $v := \dot{\varepsilon}^p$,

$$\varphi^*(\sigma) = \sup_v \left[\, \sigma v - \sigma_y|v| \,\right], \qquad \sigma v - \sigma_y|v| = |v|\left(\sigma \operatorname{sign} v - \sigma_y\right) \leq |v|\left(|\sigma| - \sigma_y\right). \tag{60}$$

If $|\sigma| \leq \sigma_y$, the argument is always nonpositive and $v = 0$ is the maximizer, so $\varphi^* = 0$. If $|\sigma| > \sigma_y$, choosing $\operatorname{sign} v = \operatorname{sign}\sigma$ and letting $|v|$ grow maximizes the dual, so $\varphi^* = +\infty$. Hence

$$\varphi^*(\sigma) = \begin{cases} 0 & |\sigma| \leq \sigma_y, \\ +\infty & |\sigma| > \sigma_y, \end{cases} \tag{61}$$

a flat floor over the elastic domain with vertical walls at its edges. The dual of a corner is a wall, and the wall stands at the extreme slopes of the corner. Its gradient vanishes in the elastic region, correctly predicting zero plastic flow. But the gradient fails on the wall itself, the one place flow happens. This is where the yield function $f(\sigma) = |\sigma| - \sigma_y$ takes over as the smooth description.

The recovered potential of the maximum-dissipation section, $\varphi(\dot{\boldsymbol{\xi}}) = \max_{f(\boldsymbol{X})\leq 0} \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}}$, is the transform of this wall taken back to the flux space, and its subdifferential at zero is the elastic domain, $\partial\varphi(\mathbf{0}) = E$. The generalized force-flux relation $\boldsymbol{X} \in \partial\varphi(\dot{\boldsymbol{\xi}})$ therefore reproduces the admissible set at rest, and at nonzero flux it yields the associative flow rule with its KKT conditions.

B. The return-mapping algorithm

The flow rules integrate in time by an implicit scheme built on the frozen-variable test of the consistency section. Given the state $(\boldsymbol{\varepsilon}_n^p, \alpha_n)$ and the new strain $\boldsymbol{\varepsilon}_{n+1}$, one load step follows.

1. Compute trial state while internal variables are frozen:

$$\boldsymbol{\sigma}^{tr} := \left.\frac{\partial\psi^e}{\partial\boldsymbol{\varepsilon}^e}\right|_{\boldsymbol{\varepsilon}^e=\boldsymbol{\varepsilon}_{n+1}-\boldsymbol{\varepsilon}_n^p}, \qquad A^{tr} := -\left.\frac{d\psi^h}{d\alpha}\right|_{\alpha=\alpha_n}. \tag{62}$$

2. If $f(\boldsymbol{\sigma}^{tr}, A^{tr}) \leq 0$, the step is elastic, accept the trial state, and set $\Delta\gamma = 0$.

3. Otherwise solve for $\Delta\gamma > 0$ and the updated state:

$$\boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \Delta\gamma \left.\frac{\partial g}{\partial\boldsymbol{\sigma}}\right|_{n+1}, \qquad \alpha_{n+1} = \alpha_n + \Delta\gamma \left.\frac{\partial g}{\partial A}\right|_{n+1}, \qquad f(\boldsymbol{\sigma}_{n+1}, A_{n+1}) = 0, \tag{63}$$

   with $\boldsymbol{\sigma}_{n+1}$ and A_{n+1} evaluated at the updated state. The last equation is the discrete consistency condition, the persistence idea of the consistency section imposed at the step's end.

For J_2 plasticity with linear hardening the system collapses to the classical radial return: the trial deviator is scaled back onto the yield surface along its own direction, so $\Delta\gamma = f(\boldsymbol{\sigma}^{tr}, A^{tr})/(3\mu+H)$.