Mechanics of Deformable Porous Media

Energy and Dissipation, Slot 2: Thermodynamics and Dissipation Potentials

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This lecture derives the constitutive equations from the two laws of thermodynamics.

1. The two laws and the dissipation inequality

Axiom: First law of thermodynamics

The internal energy of any body changes by the mechanical power done and by heating:

$$\frac{d}{dt}\int_\Omega e \, \mathrm{d}V \;=\; \mathcal{P}_{act} \;+\; \int_\Omega r \, \mathrm{d}V \;-\; \oint_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n} \, \mathrm{d}A, \tag{1}$$

with e the internal energy density, $\boldsymbol{q}$ the heat flux, and r the heat supply.

Proposition: Local form of the first law

For quasi-static processes the first law localizes to

$$\dot{e} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \nabla\cdot\boldsymbol{q} + r. \tag{2}$$

Proof. The actual power equals the external virtual power, as loads act through current velocity:

$$\mathcal{P}_{act} = \mathcal{P}_{ext} = \int_\Omega \boldsymbol{b}\cdot\dot{\boldsymbol{u}} \, \mathrm{d}V + \int_{\Gamma_t} \boldsymbol{t}\cdot\dot{\boldsymbol{u}} \, \mathrm{d}A. \tag{3}$$

At any given instant, Ω is in equilibrium, and with fixed supports $\dot{\boldsymbol{u}}$ vanishes on Γ_u. Thus, $\dot{\boldsymbol{u}}$ is a kinematically admissible variation $\delta\boldsymbol{u} = \dot{\boldsymbol{u}}$. Hence, principle of virtual work gives:

$$\mathcal{P}_{ext} = \int_\Omega \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \, \mathrm{d}V. \tag{4}$$

The divergence theorem converts the boundary heating, $\oint_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n} \, \mathrm{d}A = \int_\Omega \nabla\cdot\boldsymbol{q} \, \mathrm{d}V$. Substituting both into the first law, and noting the balance holds for any Ω, the integrand identity follows. □

Axiom: Second law of thermodynamics

The entropy density s, per unit bulk volume, satisfies

$$\dot{s} \;\geq\; -\nabla\cdot\left(\frac{\boldsymbol{q}}{T}\right) + \frac{r}{T}, \tag{5}$$

This means entropy is produced at least as fast as it is supplied by heat.

Definition: Helmholtz free energy

The Helmholtz free energy is defined as

$$\psi := e - T\,s. \tag{6}$$

The symbol is the same as the strain energy density because under isothermal elasticity, the two definitions are equivalent. This is discussed below.

Theorem: Clausius–Duhem inequality

The two laws combine into the dissipation inequality:

$$\mathcal{D} := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} - s\,\dot{T} - \frac{\boldsymbol{q}\cdot\nabla T}{T} \;\geq\; 0. \tag{7}$$

Proof. Expand $\nabla\cdot(\boldsymbol{q}/T) = (\nabla\cdot\boldsymbol{q})/T - \boldsymbol{q}\cdot\nabla T/T^2$ and multiply the second law by $T > 0$,

$$T\dot{s} \;\geq\; -\nabla\cdot\boldsymbol{q} + \frac{\boldsymbol{q}\cdot\nabla T}{T} + r. \tag{8}$$

The first law gives $-\nabla\cdot\boldsymbol{q} + r = \dot{e} - \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$, and substituting,

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \left(\dot{e} - T\dot{s}\right) - \frac{\boldsymbol{q}\cdot\nabla T}{T} \;\geq\; 0. \tag{9}$$

Differentiating the definition of ψ gives $\dot{e} - T\dot{s} = \dot{\psi} + s\dot{T}$, and substituting yields the result. □

Corollary: Isothermal dissipation inequality

For constant, uniform temperature the dissipation inequality reduces to

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} \;\geq\; 0, \tag{10}$$

the stress power not stored as free energy is dissipated. This is used in the rest of the course.

Remark: Hyperelasticity is the zero-dissipation case

If $\psi = \psi(\boldsymbol{\varepsilon})$ with $\boldsymbol{\sigma} = \partial\psi/\partial\boldsymbol{\varepsilon}$, the chain rule gives $\dot{\psi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$, so $\mathcal{D} = 0$. Every increment of work is stored in ψ and is recoverable, which is precisely the defining property of strain

energy density. For isothermal elasticity, Helmholtz and strain energy density are the same.

The next section derives constitutive relations from $\mathcal{D} \geq 0$, which is the Coleman–Noll argument.

2. The Coleman–Noll procedure

For a thermoelastic material, assume state functions depend on strain and temperature, $\psi = \psi(\boldsymbol{\varepsilon}, T)$, $s = s(\boldsymbol{\varepsilon}, T)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, T)$, and heat flux on state and gradient, $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}, T, \nabla T)$. A *process* is a time history of the state. At any instant state, and rates $(\dot{\boldsymbol{\varepsilon}}, \dot{T})$ can be assigned independently.

Theorem: Coleman–Noll

If Clausius–Duhem inequality holds for every process, then constitutive functions satisfy

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}, \qquad s = -\frac{\partial \psi}{\partial T}, \qquad -\boldsymbol{q} \cdot \nabla T \;\geq\; 0. \tag{11}$$

Proof. By chain rule $\dot{\psi} = (\partial\psi/\partial\boldsymbol{\varepsilon}) : \dot{\boldsymbol{\varepsilon}} + (\partial\psi/\partial T)\,\dot{T}$, and substituting into the dissipation inequality,

$$\mathcal{D} = \Big(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}\Big) : \dot{\boldsymbol{\varepsilon}} - \Big(s + \frac{\partial \psi}{\partial T}\Big)\dot{T} - \frac{\boldsymbol{q} \cdot \nabla T}{T} \;\geq\; 0. \tag{12}$$

At a given state, the rates can be assigned arbitrarily and independently, while the parenthesized coefficients depend only on state. Choose $\dot{T} = 0$ and $\dot{\boldsymbol{\varepsilon}} = -\lambda\big(\boldsymbol{\sigma} - \partial\psi/\partial\boldsymbol{\varepsilon}\big)$ with $\lambda > 0$. Thus,

$$-\lambda \Big(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}\Big) : \Big(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}\Big) - \frac{\boldsymbol{q} \cdot \nabla T}{T} \;\geq\; 0, \tag{13}$$

Letting $\lambda \to \infty$ forces $\boldsymbol{\sigma} = \partial\psi/\partial\boldsymbol{\varepsilon}$. Choosing instead $\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$ and $\dot{T} = -\lambda\big(s + \partial\psi/\partial T\big)$ forces $s = -\partial\psi/\partial T$ the same way. What remains is $-\boldsymbol{q} \cdot \nabla T/T \geq 0$. Hence the result is proven. □

Corollary: Maxwell relation

The stress and entropy responses of a material are linked,

$$\frac{\partial \boldsymbol{\sigma}}{\partial T} = -\frac{\partial s}{\partial \boldsymbol{\varepsilon}}. \tag{14}$$

Proof. Both sides are the mixed second derivative $\partial^2\psi/\partial T\,\partial\boldsymbol{\varepsilon}$ in opposite order. □

Now let's consider what happens if state functions depend on strain rate $\psi = \psi(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$, $s = s(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$. To simplify we assume isothermal conditions of constant and uniform temperature.

Proposition: Rate-dependent stress

If $\psi = \psi(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})$, the dissipation inequality implies

$$\frac{\partial \psi}{\partial \dot{\boldsymbol{\varepsilon}}} = \mathbf{0}, \qquad \boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^v, \qquad \boldsymbol{\sigma}^v : \dot{\boldsymbol{\varepsilon}} \ \geq \ 0, \tag{15}$$

The interpretation is that free energy cannot store rates, and the stress splits into an elastic and a viscous stress $\boldsymbol{\sigma}^v$. The above Maxwell relation fails for this rate-dependent stress.

Proof. The chain rule gives $\dot{\psi} = (\partial\psi/\partial\boldsymbol{\varepsilon}) : \dot{\boldsymbol{\varepsilon}} + (\partial\psi/\partial\dot{\boldsymbol{\varepsilon}}) : \ddot{\boldsymbol{\varepsilon}}$, so

$$\mathcal{D} = \left(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \psi}{\partial \dot{\boldsymbol{\varepsilon}}} : \ddot{\boldsymbol{\varepsilon}} \ \geq \ 0. \tag{16}$$

The strain, its rate, and acceleration are independently assignable, but only $\ddot{\boldsymbol{\varepsilon}}$ has a coefficient free of $\ddot{\boldsymbol{\varepsilon}}$. Picking $\ddot{\boldsymbol{\varepsilon}} = \lambda\, \partial\psi/\partial\dot{\boldsymbol{\varepsilon}}$ and letting $\lambda \to \infty$ forces $\partial\psi/\partial\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$. Defining $\boldsymbol{\sigma}^v := \boldsymbol{\sigma} - \partial\psi/\partial\boldsymbol{\varepsilon}$, what remains is $\boldsymbol{\sigma}^v : \dot{\boldsymbol{\varepsilon}} \geq 0$. The scaling argument does not apply to $\boldsymbol{\sigma}^v$ because it depends on $\dot{\boldsymbol{\varepsilon}}$. □

Remark: What the second law constrains

The second law acts as a filter on constitutive laws. Any admissible material model must satisfy the dissipation inequality, for every conceivable process. Coleman–Noll is the key tool for developing constitutive relations and dissipation potentials, which we discuss next.

3. Internal variables and the dissipation potential

Elasticity stores all of the work done on the system. However, real materials also flow, yield, and break, which dissipate energy. From here onward, we assume isothermal conditions for simplicity.

Definition: Internal variables

The constitutive functions are augmented with internal variables $\boldsymbol{\xi}$ to capture the material's state beyond strain, $\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{\xi})$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\xi})$. Internal variables are states, not rates. Examples include plastic strain $\boldsymbol{\varepsilon}^p$ and damage variable d, which record irreversible changes. Unlike strain, internal variables cannot be controlled externally by imposed loads or displacements. They evolve via their own laws, compatible with the dissipation inequality.

Proposition: Isothermal dissipation inequality

The isothermal dissipation inequality with $\psi = \psi(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, \boldsymbol{\xi})$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, \boldsymbol{\xi})$ is equivalent to

$$\frac{\partial \psi}{\partial \dot{\boldsymbol{\varepsilon}}} = \mathbf{0}, \qquad \boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^v, \qquad \mathcal{D} = \boldsymbol{\sigma}^v : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \ \geq \ 0, \tag{17}$$

where $\boldsymbol{\sigma}^v$ is viscous stress and $\boldsymbol{X} := -\partial\psi/\partial\boldsymbol{\xi}$ is the thermodynamic force conjugate to $\boldsymbol{\xi}$.

Proof. The chain rule gives

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \psi}{\partial \dot{\boldsymbol{\varepsilon}}} : \ddot{\boldsymbol{\varepsilon}} + \frac{\partial \psi}{\partial \boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}}. \tag{18}$$

The $\ddot{\boldsymbol{\varepsilon}}$ term is killed exactly as in the rate-dependent stress proposition, so $\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{\xi})$. Substituting into the isothermal dissipation inequality and defining $\boldsymbol{\sigma}^v := \boldsymbol{\sigma} - \partial\psi/\partial\boldsymbol{\varepsilon}$,

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} = \boldsymbol{\sigma}^v : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{X} \cdot \dot{\boldsymbol{\xi}} \;\geq\; 0. \tag{19}$$

The $\boldsymbol{\sigma}^v$ depends on $\dot{\boldsymbol{\varepsilon}}$ itself, so the scaling argument does not apply, and $\dot{\boldsymbol{\xi}}$ is not freely assignable as it cannot be controlled externally. Therefore, the expression cannot be reduced further. □

Definition: Dissipation potential

Suppose the dissipation contains a product $\boldsymbol{F} \cdot \boldsymbol{J}$, with $\boldsymbol{J}$ a rate or flux and $\boldsymbol{F}$ its conjugate force. A dissipation potential for the pair is defined as a convex function $\phi(\boldsymbol{J}) \geq 0$ with $\phi(\boldsymbol{0}) = 0$. Hence, ϕ has a global minimum at $\boldsymbol{J} = \boldsymbol{0}$ and induces the constitutive law

$$\boldsymbol{F} = \frac{\partial \phi}{\partial \boldsymbol{J}}. \tag{20}$$

The proposition above supplies two such pairs, $(\dot{\boldsymbol{\xi}}, \boldsymbol{X})$ and $(\dot{\boldsymbol{\varepsilon}}, \boldsymbol{\sigma}^v)$. Transport supplies two more, which are the heat and fluid fluxes with their gradient forces derived in the next sections. Each pair has its own potential, and the theorem below makes every product nonnegative. For internal variables, the constitutive law balances the energetic force against the dissipative force,

$$-\frac{\partial \psi}{\partial \boldsymbol{\xi}} = \frac{\partial \phi}{\partial \dot{\boldsymbol{\xi}}}. \tag{21}$$

Theorem: Convexity guarantees the second law

Every product generated by a dissipation potential is nonnegative,

$$\boldsymbol{F} \cdot \boldsymbol{J} = \frac{\partial \phi}{\partial \boldsymbol{J}} \cdot \boldsymbol{J} \;\geq\; \phi(\boldsymbol{J}) \;\geq\; 0. \tag{22}$$

Proof. A smooth convex function lies above every tangent, $\phi(\boldsymbol{y}) \geq \phi(\boldsymbol{J}) + \partial\phi/\partial\boldsymbol{J} \cdot (\boldsymbol{y} - \boldsymbol{J})\ \forall\, \boldsymbol{y}$. Setting $\boldsymbol{y} = \boldsymbol{0}$ and using $\phi(\boldsymbol{0}) = 0$ gives $\partial\phi/\partial\boldsymbol{J} \cdot \boldsymbol{J} \geq \phi(\boldsymbol{J})$, and $\phi(\boldsymbol{J}) \geq 0$ holds from definition. □

Example: Kelvin–Voigt viscoelastic solid

Take $\psi = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}$ with no internal variables, and generate the viscous stress from the dissipation potential $\phi(\dot{\boldsymbol{\varepsilon}}) = \frac{1}{2}\,\eta\,\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}$, so $\boldsymbol{\sigma}^v = \partial\phi/\partial\dot{\boldsymbol{\varepsilon}} = \eta\,\dot{\boldsymbol{\varepsilon}}$ and the total stress is

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} + \eta\,\dot{\boldsymbol{\varepsilon}}, \tag{23}$$

with $\mathcal{D} = \eta\,\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \geq 0$, positive whenever the material flows.

4. Fourier and Darcy from quadratic potentials

Transport dissipates energy through flux–force pairs. Coleman–Noll already produced the thermal pair, $(\boldsymbol{q}, -\nabla T/T)$. A porous medium adds the fluid flux $\boldsymbol{w}$ with force $-\nabla p$, derived next.

Proposition: Quadratic potentials generate linear laws

For a vector flux $\boldsymbol{J}$ with conjugate force $\boldsymbol{F}$, the quadratic dissipation potential

$$\phi(\boldsymbol{J}) = \tfrac{1}{2}\,\boldsymbol{J}\cdot\boldsymbol{M}^{-1}\boldsymbol{J}, \tag{24}$$

with $\boldsymbol{M}$ denoting a symmetric positive definite mobility tensor, generates the linear law $\boldsymbol{J} = \boldsymbol{M}\,\boldsymbol{F}$. Moreover, the product $\boldsymbol{F}\cdot\boldsymbol{J} = \boldsymbol{J}\cdot\boldsymbol{M}^{-1}\boldsymbol{J} > 0$ for $\boldsymbol{J} \neq \boldsymbol{0}$.

Proof. Differentiating the potential gives $\boldsymbol{F} = \partial\phi/\partial\boldsymbol{J} = \boldsymbol{M}^{-1}\boldsymbol{J}$, and inverting gives $\boldsymbol{J} = \boldsymbol{M}\,\boldsymbol{F}$. Substituting this into $\boldsymbol{F}\cdot\boldsymbol{J}$ gives $\boldsymbol{J}\cdot\boldsymbol{M}^{-1}\boldsymbol{J}$, which is positive because $\boldsymbol{M}$ is positive definite. □

For tensor-valued rates, the mobility becomes a fourth-order tensor with major symmetry, like $\mathbb{C}$.

Example: Fourier conduction

Take $\boldsymbol{J} = \boldsymbol{q}$ and $\boldsymbol{F} = -\nabla T/T$ with mobility $\boldsymbol{M} = T\,\boldsymbol{k}$, where $\boldsymbol{k}$ is the symmetric positive definite conductivity tensor. The dissipation potential and the resulting linear law are

$$\phi(\boldsymbol{q}) = \frac{1}{2T}\,\boldsymbol{q}\cdot\boldsymbol{k}^{-1}\boldsymbol{q}, \qquad \boldsymbol{q} = -\,\boldsymbol{k}\,\nabla T, \tag{25}$$

which is Fourier's law. The contributing term in the dissipation then reads

$$\boldsymbol{F}\cdot\boldsymbol{J} = \frac{\nabla T\cdot\boldsymbol{k}\,\nabla T}{T} \;\geq\; 0. \tag{26}$$

which implies that conduction produces entropy.

Example: Darcy flow

Take $\boldsymbol{J} = \boldsymbol{w}$ to be the Darcy flux and $\boldsymbol{F} = -\nabla p$ with mobility $\boldsymbol{M} = \boldsymbol{\kappa}/\eta_f$, where $\boldsymbol{\kappa}$ is the symmetric positive definite permeability tensor and $\eta_f > 0$ is fluid viscosity. Thus,

$$\phi(\boldsymbol{w}) = \frac{\eta_f}{2}\,\boldsymbol{w}\cdot\boldsymbol{\kappa}^{-1}\boldsymbol{w}, \qquad \boldsymbol{w} = -\,\frac{1}{\eta_f}\,\boldsymbol{\kappa}\,\nabla p, \tag{27}$$

which is known as Darcy's law. The contributing term in the dissipation inequality is

$$\boldsymbol{F}\cdot\boldsymbol{J} = \frac{\nabla p\cdot\boldsymbol{\kappa}\,\nabla p}{\eta_f} \;\geq\; 0. \tag{28}$$

Similar to Fourier's law, Darcy's law is entropy producing.

The symmetry of $\boldsymbol{k}$ and $\boldsymbol{\kappa}$ is the statement of Onsager reciprocity, discussed later in the course.

5. Poroelasticity: the Biot energy and effective stress

Definition: Fluid content

For a fluid-saturated porous element, the fluid content ζ is the volume of fluid exchanged with the element per unit bulk volume. By convention, it is positive for fluid entering.

Proposition: Dissipation inequality of a saturated medium

For isothermal conditions in a porous medium, the dissipation inequality reads

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + p\,\dot{\zeta} - \dot{\psi} \;-\; \boldsymbol{w}\cdot\nabla p \;\geq\; 0, \tag{29}$$

where fluid storage and flow contribute as separate terms.

Proof. The isothermal second law states that any unstored power gets dissipated. The surroundings supply power in two ways. The first is the mechanical power, which by the principle of virtual work with $\delta\boldsymbol{u} = \dot{\boldsymbol{u}}$ equals the internal power, $\int_\Omega \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}V$. Recall the principle of virtual work is equivalent to linear momentum balance, which accounts for fluid pressure through the total stress $\boldsymbol{\sigma}$. The second is the work of pushing fluid across the boundary against the pore pressure:

$$-\oint_{\partial\Omega} p\,\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}A = \int_\Omega \left(p\,\dot{\zeta} - \boldsymbol{w}\cdot\nabla p\right)\,\mathrm{d}V, \tag{30}$$

This equation is derived by using the divergence theorem, the $\nabla\cdot(p\,\boldsymbol{w}) = p\,\nabla\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\nabla p$ identity, and the fluid mass balance equation $\dot{\zeta} + \nabla\cdot\boldsymbol{w} = 0$. Collecting both power contributions,

$$\int_\Omega \left(\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + p\,\dot{\zeta} - \boldsymbol{w}\cdot\nabla p - \dot{\psi}\right)\,\mathrm{d}V \;\geq\; 0, \tag{31}$$

and since this holds for any Ω, the local inequality follows. □

Proposition: Poroelastic constitutive relations

If $\psi = \psi(\boldsymbol{\varepsilon}, \zeta)$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \zeta)$, the dissipation inequality implies

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}, \qquad p = \frac{\partial\psi}{\partial\zeta}, \qquad -\boldsymbol{w}\cdot\nabla p \;\geq\; 0. \tag{32}$$

Storage does not dissipate energy, and flow-induced entropy production is via $(\boldsymbol{w}, -\nabla p)$.

Proof. The chain rule gives

$$\dot{\psi} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial\psi}{\partial\zeta}\,\dot{\zeta}. \tag{33}$$

Substituting into (29) and collecting terms,

$$\mathcal{D} = \left(\boldsymbol{\sigma} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} + \left(p - \frac{\partial\psi}{\partial\zeta}\right)\dot{\zeta} \;-\; \boldsymbol{w}\cdot\nabla p \;\geq\; 0. \tag{34}$$

Because we can control $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\zeta}$ independently, $\dot{\boldsymbol{\varepsilon}}$ by the testing machine and $\dot{\zeta}$ by a pump, their multiplying factors must vanish by the Coleman–Noll argument. What remains is $-\boldsymbol{w}\cdot\nabla p \geq 0$. □

Example: Biot's free energy

The isotropic linear poroelastic free energy is

$$\psi(\boldsymbol{\varepsilon},\zeta) = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} + \tfrac{1}{2}\,M\big(\zeta - b\,\varepsilon_v\big)^2, \tag{35}$$

with the drained stiffness $\mathbb{C}$, the Biot modulus $M > 0$, and the Biot coefficient b. Differentiating, with $\partial\varepsilon_v/\partial\boldsymbol{\varepsilon} = \mathbf{1}$, we get

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} - b\,p\,\mathbf{1}, \qquad p = M\big(\zeta - b\,\varepsilon_v\big). \tag{36}$$

Effective stress is defined by $\boldsymbol{\sigma}' := \boldsymbol{\sigma} + b\,p\,\mathbf{1} = \mathbb{C} : \boldsymbol{\varepsilon}$ and is governed by Hooke's law.

Corollary: Maxwell relation

The following identity holds,

$$\frac{\partial\boldsymbol{\sigma}}{\partial\zeta} = \frac{\partial p}{\partial\boldsymbol{\varepsilon}} = -\,b\,M\,\mathbf{1}, \tag{37}$$

where the rightmost equality is valid for linear poroelasticity.

Proof. Both sides are the mixed second derivative of ψ. □

6. The meaning of b and M

Two tests pin the constants. A drained (jacketed) test encloses the sample in an impermeable jacket with a drainage line holding $p = 0$, so the stress law gives $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$, and $\mathbb{C}$ with its bulk modulus K are the drained moduli. An unjacketed test immerses the bare sample in fluid at pressure p. The fluid penetrates the pores, so the same pressure p loads every external and internal surface alike.

Proposition: The Biot coefficient

Let K_s denote the bulk modulus of the solid grains, then

$$b = 1 - \frac{K}{K_s}. \tag{38}$$

Proof. In the unjacketed test fluid pushes on the outer sample surface, so the total stress is $\boldsymbol{\sigma} = -p\,\mathbf{1}$, and pore pressure equals p. Every grain is loaded by the same pressure on all surfaces, so the whole sample compresses homogeneously as the solid grains,

$$\varepsilon_v = -\,\frac{p}{K_s}. \tag{39}$$

The volumetric part of the stress relation, $\sigma_m = K\,\varepsilon_v - b\,p$ (recall $\sigma_m = \mathrm{tr}(\boldsymbol{\sigma})/3$), then reads

$$-\,p = -\,K\,\frac{p}{K_s} - b\,p, \tag{40}$$

Dividing by $-p$ gives the result. □

Proposition: The Biot modulus

Let ϕ_f denote porosity and K_f the fluid's bulk modulus, then

$$\frac{1}{M} = \frac{\phi_f}{K_f} + \frac{b-\phi_f}{K_s}. \tag{41}$$

Proof. Consider the unjacketed test again. The sample compresses homogeneously, thus the pore space shrinks at the same relative rate as the bulk,

$$\frac{\Delta V_{pore}}{V_{pore}} = \varepsilon_v = -\,\frac{p}{K_s}. \tag{42}$$

The fluid occupying the pores, if none entered, would shrink by $\Delta V_f/V_f = -p/K_f$. This is more than the amount pores shrink if $K_f < K_s$. Therefore, keeping the pores saturated requires fluid entry by an amount equal to the deficit created. This equals $p/K_f - p/K_s$ on a per-unit-pore-volume basis. Multiplying this by porosity ϕ_f converts this to per-unit-bulk-volume basis,

$$\zeta = \phi_f\,p\left(\frac{1}{K_f} - \frac{1}{K_s}\right). \tag{43}$$

Independently, solving the pressure relation, $p = M(\zeta - b\,\varepsilon_v)$, for ζ with $\varepsilon_v = -p/K_s$ gives

$$\zeta = \frac{p}{M} + b\,\varepsilon_v = \frac{p}{M} - b\,\frac{p}{K_s}. \tag{44}$$

Equating the above two expressions for ζ and dividing by p yields the result. □

Remark: Terzaghi limit

For soils, grains are nearly incompressible relative to the porous skeleton. Thus, $K_s \to \infty$, and $b \to 1$ holds. Moreover, the effective stress reduces to $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\,\mathbf{1}$, which is the form Terzaghi had postulated. Consolidated rocks exhibit $b < 1$.