Mechanics of Deformable Porous Media

Energy and Dissipation, Slot 1: Virtual Work and Energy Theorems

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This lecture converts the variational machinery of the previous lecture into the working energy theorems of mechanics.

1. Work and strain energy

Two different objects are called work, and confusing them causes errors. Let's separate them.

Definition: Thermodynamic work

The thermodynamic work is the work actually done on the body during loading,

$$W_{act} = \int_{\text{loading path}} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{u}, \tag{1}$$

a path integral over the loading history.

Definition: Load work functional

For fixed loads $\boldsymbol{b}$ in Ω and $\bar{\boldsymbol{t}}$ on Γ_t,

$$W_{ext}[\boldsymbol{u}] = \int_\Omega \boldsymbol{b} \cdot \boldsymbol{u} \, \mathrm{d}V + \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{u} \, \mathrm{d}A, \tag{2}$$

the work the final loads do if they acted at full strength through the displacement $\boldsymbol{u}$. This is a functional of $\boldsymbol{u}$, not a path integral. It is used in the total potential energy, $\Pi[\boldsymbol{u}] = U - W_{ext}$.

Example. A linear spring of stiffness k loaded quasi-statically to final displacement u_f carries the force $f = ku$ that grows from zero. Therefore, $W_{act} = \int_0^{u_f} ku \, \mathrm{d}u = \frac{1}{2}ku_f^2$ which is the area of the triangle under the force–displacement line. The load work is $W_{ext} = f_f \, u_f = ku_f^2$, corresponding to a rectangle. The factor of two is the entire difference between the two notions.

Definition: Strain energy functional

Given a hyperelastic material whose stress derives from a strain energy density, the strain energy functional is

$$U[\boldsymbol{u}] = \int_\Omega \psi\big(\boldsymbol{\varepsilon}(\boldsymbol{u})\big)\ \mathrm{d}V, \qquad \boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}. \tag{3}$$

Example. For linear elasticity $\psi = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$ hold. Moreover, recall the existence of ψ was shown to be equivalent to the major symmetry of $\mathbb{C}$, as proved in the first lecture.

Axiom: First law of thermodynamics (adiabatic)

For quasi-static loading with no heat exchange, the work done is stored as strain energy,

$$U = W_{act}. \tag{4}$$

2. Total potential energy and the principle of virtual work

Definition: Admissible virtual displacements

A displacement field is kinematically admissible if $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_u. A virtual displacement $\delta\boldsymbol{u}$ is a variation of an admissible field, so $\delta\boldsymbol{u} = \boldsymbol{0}$ on Γ_u, with virtual strain $\delta\boldsymbol{\varepsilon} := \frac{1}{2}\big(\nabla\delta\boldsymbol{u} + \nabla\delta\boldsymbol{u}^\top\big)$.

Definition: Total potential energy

The total potential energy of a body is defined as

$$\Pi[\boldsymbol{u}] = U[\boldsymbol{u}] - W_{ext}[\boldsymbol{u}] = \int_\Omega \psi(\boldsymbol{\varepsilon})\ \mathrm{d}V - \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}\ \mathrm{d}V - \int_{\Gamma_t} \bar{\boldsymbol{t}}\cdot\boldsymbol{u}\ \mathrm{d}A, \tag{5}$$

over kinematically admissible displacements.

Theorem: Principle of virtual work

Stationarity of $\Pi[\boldsymbol{u}]$ over admissible displacements is equivalent to the *weak form*,

$$\int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\ \mathrm{d}V = \int_\Omega \boldsymbol{b}\cdot\delta\boldsymbol{u}\ \mathrm{d}V + \int_{\Gamma_t} \bar{\boldsymbol{t}}\cdot\delta\boldsymbol{u}\ \mathrm{d}A \qquad \forall \text{ admissible } \delta\boldsymbol{u}, \tag{6}$$

which balances internal and external virtual work. This is also equivalent to the *strong form*,

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \ \text{ in } \Omega, \qquad \boldsymbol{\sigma}\,\boldsymbol{n} = \bar{\boldsymbol{t}} \ \text{ on } \Gamma_t. \tag{7}$$

which is the Euler–Lagrange equation of $\Pi[\boldsymbol{u}]$ with natural boundary condition on Γ_t.

Proof. Strain is linear in displacement, so $\delta\psi = (\partial\psi/\partial\boldsymbol{\varepsilon}) : \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}$. Hence, by the δ rule,

$$\delta\Pi = \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\ \mathrm{d}V - \int_\Omega \boldsymbol{b}\cdot\delta\boldsymbol{u}\ \mathrm{d}V - \int_{\Gamma_t} \bar{\boldsymbol{t}}\cdot\delta\boldsymbol{u}\ \mathrm{d}A, \tag{8}$$

Thus, $\delta\Pi = 0$ for all admissible $\delta\boldsymbol{u}$ yields the weak form. Next we show the weak form and strong form are equivalent. By the orthogonality lemma of the first lecture and divergence theorem,

$$\int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\ \mathrm{d}V = \int_\Omega \boldsymbol{\sigma} : \nabla\delta\boldsymbol{u}\ \mathrm{d}V = \int_{\partial\Omega} (\boldsymbol{\sigma}\,\boldsymbol{n})\cdot\delta\boldsymbol{u}\ \mathrm{d}A - \int_\Omega (\nabla\cdot\boldsymbol{\sigma})\cdot\delta\boldsymbol{u}\ \mathrm{d}V. \tag{9}$$

Inserting this into the weak form and using $\delta\boldsymbol{u} = \boldsymbol{0}$ on Γ_u yields

$$\int_\Omega \big(\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b}\big)\cdot\delta\boldsymbol{u}\ \mathrm{d}V + \int_{\Gamma_t} \big(\bar{\boldsymbol{t}} - \boldsymbol{\sigma}\,\boldsymbol{n}\big)\cdot\delta\boldsymbol{u}\ \mathrm{d}A = 0 \qquad \forall \text{ admissible } \delta\boldsymbol{u}. \tag{10}$$

Therefore, if the weak form holds, we can first pick $\delta\boldsymbol{u}$ such that it vanishes on Γ_t, which yields $\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$ in Ω. The volume term then drops for every arbitrary $\delta\boldsymbol{u}$ nonzero on Γ_t. The traction condition then follows, and the strong form holds. Conversely, if the strong form holds, both parentheses vanish, so (10) holds, and reversing the steps recovers the weak form. □

Theorem: Clapeyron's relation

For a linear elastic body in equilibrium with homogeneous essential BCs, $\bar{\boldsymbol{u}} = \boldsymbol{0}$ on Γ_u,

$$U = \tfrac{1}{2}\, W_{ext}. \tag{11}$$

Proof. With $\bar{\boldsymbol{u}} = \boldsymbol{0}$ the equilibrium field $\boldsymbol{u}$ is itself an admissible virtual displacement. Choosing $\delta\boldsymbol{u} = \boldsymbol{u}$ in the weak form gives $\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\ \mathrm{d}V = W_{ext}$, and linear elasticity gives $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = 2\psi$, hence $2U = W_{ext}$. □

With the first law, $W_{act} = U = \frac{1}{2}W_{ext}$, which is the general form of the triangle argument before.

Theorem: Minimum total potential energy

Let $\mathbb{C} := \partial\boldsymbol{\sigma}/\partial\boldsymbol{\varepsilon}$ be positive definite, $\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} > 0\ \forall\, \boldsymbol{\varepsilon} \neq \boldsymbol{0}$, and let Γ_u have nonzero area. Then the stationary point $\boldsymbol{u}$ is the unique global minimizer,

$$\Pi[\boldsymbol{u} + \delta\boldsymbol{u}] = \Pi[\boldsymbol{u}] + \tfrac{1}{2}\int_\Omega \delta\boldsymbol{\varepsilon} : \mathbb{C} : \delta\boldsymbol{\varepsilon}\ \mathrm{d}V \ > \ \Pi[\boldsymbol{u}] \qquad \forall \text{ admissible } \delta\boldsymbol{u} \neq \boldsymbol{0}. \tag{12}$$

Proof. The first variation is the weak form and vanishes by stationarity,

$$\delta\Pi = \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\ \mathrm{d}V - \int_\Omega \boldsymbol{b}\cdot\delta\boldsymbol{u}\ \mathrm{d}V - \int_{\Gamma_t} \bar{\boldsymbol{t}}\cdot\delta\boldsymbol{u}\ \mathrm{d}A = 0. \tag{13}$$

Applying δ again and recognizing $\delta\boldsymbol{\sigma} = \mathbb{C} : \delta\boldsymbol{\varepsilon}$ and $\delta^2\boldsymbol{\varepsilon} = \boldsymbol{0}$, we obtain

$$\delta^2\Pi = \int_\Omega \delta\boldsymbol{\varepsilon} : \mathbb{C} : \delta\boldsymbol{\varepsilon}\ \mathrm{d}V. \tag{14}$$

A third application of δ yields $\delta^3\Pi = 0$ because $\mathbb{C}$ is constant. The Taylor expansion therefore terminates at the second-order term, giving the stated identity. The remainder is positive by the positive definiteness of $\mathbb{C}$. This is unless $\delta\boldsymbol{\varepsilon} \equiv \mathbf{0}$, which makes $\delta\boldsymbol{u}$ an infinitesimal rigid motion, $\delta\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{\omega} \times \boldsymbol{x}$, and a rigid motion vanishing on Γ_u of nonzero area is identically zero. □

Remark: The weak form and finite elements

The finite element method discretizes the weak form of the boundary value problem, whereas finite difference and finite volume methods discretize the strong form of the equations.

3. Reciprocity and Castigliano's theorems

Theorem: Betti's reciprocity

Let two load systems $(\boldsymbol{b}^{(1)}, \bar{\boldsymbol{t}}^{(1)})$ and $(\boldsymbol{b}^{(2)}, \bar{\boldsymbol{t}}^{(2)})$ act on the same linear elastic body with the same homogeneous essential BCs, $\boldsymbol{u} = \mathbf{0}$ on Γ_u, producing $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$. Then the virtual work of the first loads through the second displacements equals the virtual work of the second loads through the first,

$$\int_\Omega \boldsymbol{b}^{(1)} \cdot \boldsymbol{u}^{(2)}\,\mathrm{d}V + \int_{\Gamma_t} \bar{\boldsymbol{t}}^{(1)} \cdot \boldsymbol{u}^{(2)}\,\mathrm{d}A \;=\; \int_\Omega \boldsymbol{b}^{(2)} \cdot \boldsymbol{u}^{(1)}\,\mathrm{d}V + \int_{\Gamma_t} \bar{\boldsymbol{t}}^{(2)} \cdot \boldsymbol{u}^{(1)}\,\mathrm{d}A. \tag{15}$$

Proof. With homogeneous essential boundary conditions, $\boldsymbol{u}^{(2)}$ is an admissible virtual displacement for system 1. The weak form of system 1 tested with $\delta\boldsymbol{u} = \boldsymbol{u}^{(2)}$ gives $\int_\Omega \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon}^{(2)}\,\mathrm{d}V = W_{12}$, and symmetrically $\int_\Omega \boldsymbol{\sigma}^{(2)} : \boldsymbol{\varepsilon}^{(1)}\,\mathrm{d}V = W_{21}$. The major symmetry gives

$$\boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{(1)} : \mathbb{C} : \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{(2)} : \mathbb{C} : \boldsymbol{\varepsilon}^{(1)} = \boldsymbol{\sigma}^{(2)} : \boldsymbol{\varepsilon}^{(1)}, \tag{16}$$

so $W_{12} = W_{21}$. □

Reciprocity is major symmetry integrated over the body. In discretized form it is $\boldsymbol{K} = \boldsymbol{K}^\top$.

Definition: Complementary energy

For a structure loaded by forces $P_1, \ldots, P_n$, with the resulting (conjugate) displacements $u_1, \ldots, u_n$ at load points and strain energy $U(u_1, \ldots, u_n)$, the complementary energy is defined as the Legendre transform

$$U^*(P_1, \ldots, P_n) := \sum_i P_i\, u_i - U, \tag{17}$$

with the u_i eliminated through $P_i = \partial U/\partial u_i$ (using Castigliano's relation I below).

Theorem: Castigliano's relations I and II

At equilibrium, the following hold

$$(\mathrm{I}) \quad P_i = \frac{\partial U}{\partial u_i}, \qquad\qquad (\mathrm{II}) \quad u_i = \frac{\partial U^*}{\partial P_i}. \tag{18}$$

Proof. (I) The total potential energy is $\Pi = U(u_1, \ldots, u_n) - \sum_i P_i u_i$, and stationarity in each u_i gives $\partial U/\partial u_i = P_i$. (II) Differentiating the definition of U^*,

$$\frac{\partial U^*}{\partial P_i} = u_i + \sum_j P_j \frac{\partial u_j}{\partial P_i} - \sum_j \frac{\partial U}{\partial u_j}\frac{\partial u_j}{\partial P_i} = u_i, \tag{19}$$

where the last two sums cancel by Castigliano's relation I. □

Definition: Linear structure

A structure is linear if the load–displacement relation is linear,

$$P_i = \sum_j K_{ij}\, u_j, \tag{20}$$

with a constant stiffness matrix $\boldsymbol{K}$, symmetric by Betti's reciprocity. A structure is linear whenever the material is linear elastic and the deformations are small.

Proposition: Complementary energy of linear structure

For a linear structure, the strain energy is the quadratic $U = \frac{1}{2}\sum_{ij} K_{ij}\, u_i\, u_j = \frac{1}{2}\sum_i P_i\, u_i$, and the complementary energy equals it *at equilibrium*,

$$U^* = U, \tag{21}$$

Thus strain energy expressed as a function of loads is the complementary energy.

Proof. Load the structure proportionally, $u_i(t) = t\, u_i$ with t going from 0 to 1, so the forces are $P_i(t) = t \sum_j K_{ij}\, u_j$. The work done is

$$W_{act} = \sum_i \int_0^1 P_i(t)\, u_i \,\mathrm{d}t = \sum_{ij} K_{ij}\, u_i\, u_j \int_0^1 t \,\mathrm{d}t = \tfrac{1}{2}\sum_{ij} K_{ij}\, u_i\, u_j, \tag{22}$$

and $U = W_{act}$ by first law. Then $\sum_i P_i\, u_i = \sum_{ij} K_{ij}\, u_i\, u_j = 2U$, and $U^* = \sum_i P_i\, u_i - U = U$. □

Example: Cantilever tip deflection

A cantilever clamped at $x = 0$ with bending stiffness EI carries a tip load P at $x = L$. Moment equilibrium of the segment $[x, L]$ determines the internal bending moment from statics alone. The only force on the segment is P at distance $L - x$ (origin at x),

$$M(x) = P\,(L - x), \tag{23}$$

the same internal moment that Euler–Bernoulli kinematics writes as $M = EI\,w''$. Substituting $w'' = M/EI$ into the beam strain energy of the previous lecture, followed by (23)

$$U = \int_0^L \tfrac{1}{2}\,EI\,(w'')^2\,\mathrm{d}x = \int_0^L \frac{M^2}{2EI}\,\mathrm{d}x = \frac{P^2L^3}{6EI}, \tag{24}$$

Note this is a function of the P only. The cantilever is a linear structure, due to elastic material at small deflection. Thus by the above proposition, $U^* = U$, and applying Castigliano's relation II gives the tip deflection without having to solve the beam equation,

$$u_{tip} = \frac{\partial U^*}{\partial P} = \frac{PL^3}{3EI}. \tag{25}$$