Mechanics of Deformable Porous Media

Fundamentals, Slot 2: Calculus of Variations

Yashar Mamani

InterPore Academy Short Course, August 2026

This lecture turns energy functionals into differential equations and boundary conditions.

1. Functionals and the first variation

Definition: Integral functional

A functional assigns a number to a function,

$$J[u] = \int_0^L F\big(x,\, u,\, u'\big)\, \mathrm{d}x. \tag{1}$$

Example. Arc length has $F = \sqrt{1+(u')^2}$. The total potential energy of a loaded bar has $F = \frac{1}{2}\, EA\,(u')^2 - f\,u$, strain energy density plus the load potential $-fu$.

Definition: Variation

A variation is a function η vanishing wherever u is prescribed, so that $u+\epsilon\eta$ remains admissible $\forall\,\epsilon$. We write $\delta u := \epsilon\,\eta$, and the first variation of J is

$$\delta J := \frac{d}{d\epsilon}\, J[u+\epsilon\eta]\,\Big|_{\epsilon=0}. \tag{2}$$

Stationarity means $\delta J = 0$ for every admissible η.

Theorem: First variation of an integral functional

The first variation of the integral functional $J[u] = \int_0^L F(x,u,u')\,\mathrm{d}x$, with $x\in[0,L]$, is

$$\delta J = \int_0^L \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right]\eta\;\mathrm{d}x \;+\; \left[\frac{\partial F}{\partial u'}\,\eta\right]_0^L. \tag{3}$$

Proof. Fix an admissible η and set $\Phi(\epsilon) := J[u+\epsilon\eta]$, so that by definition $\delta J = \Phi'(0)$. Writing out the integrand,

$$\Phi(\epsilon) = \int_0^L F\big(x,\ u + \epsilon\eta,\ u' + \epsilon\eta'\big)\ \mathrm{d}x. \tag{4}$$

The integrand and its ϵ derivative are continuous on the compact interval $[0, L]$, so we may differentiate under the integral sign. The chain rule gives

$$\Phi'(\epsilon) = \int_0^L \left[\frac{\partial F}{\partial u}\big(x,\ u + \epsilon\eta,\ u' + \epsilon\eta'\big)\,\eta + \frac{\partial F}{\partial u'}\big(x,\ u + \epsilon\eta,\ u' + \epsilon\eta'\big)\,\eta'\right] \mathrm{d}x, \tag{5}$$

and evaluating at $\epsilon = 0$,

$$\delta J = \int_0^L \Big(F_u\,\eta + F_{u'}\,\eta'\Big)\ \mathrm{d}x. \tag{6}$$

Integrating the second term by parts,

$$\int_0^L F_{u'}\,\eta'\ \mathrm{d}x = \Big[F_{u'}\,\eta\Big]_0^L - \int_0^L \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big)\,\eta\ \mathrm{d}x, \tag{7}$$

and substituting gives the stated formula. □

Everything in this lecture flows from this one formula. The interior bracket will become the differential equation, the boundary term will become the boundary conditions.

2. The fundamental lemma and the Euler–Lagrange equation

Lemma: Fundamental lemma of the calculus of variations

If g is continuous on $[0, L]$ and $\int_0^L g\,\eta\ \mathrm{d}x = 0$ for every smooth η with $\eta(0) = \eta(L) = 0$, then $g(x) = 0\ \forall\, x \in [0, L]$.

Proof. Suppose $g(x_0) \neq 0$, say $g(x_0) > 0$. Continuity gives an interval $(a, b) \ni x_0$ on which $g > 0$. Choose a smooth bump η that is positive inside (a, b) and zero elsewhere. Then $\int_0^L g\,\eta\ \mathrm{d}x > 0$, a contradiction. □

Remark: The test space stays rich

The lemma survives shrinking the test class. One may demand that any number of derivatives of η, or even all of them, also vanish at the endpoints, because bumps such as $\eta = \exp\big(-1/((x-a)(b-x))\big)$ on (a, b), zero elsewhere, are smooth, positive inside, and vanish at a and b together with every derivative. This freedom justifies the two-step proof of the natural boundary condition and the parent-existence arguments of the next section.

Theorem: Euler–Lagrange equation

If u makes J stationary among all variations with $\eta(0) = \eta(L) = 0$, then

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} = 0 \quad \text{on } (0, L). \tag{8}$$

Proof. The boundary term of the first-variation theorem vanishes because η vanishes at both ends, and the fundamental lemma applied to the interior bracket gives the result. ☐

Example: Axially loaded bar

Take the following energy functional as our running example,

$$\Pi[u] = \int_0^L \Big(\tfrac{1}{2}\, EA\,(u')^2 - f\,u\Big)\,\mathrm{d}x, \qquad u(0)=0, \quad u(L)=\bar{u}. \tag{9}$$

Both boundary conditions are Dirichlet, so admissible variations vanish at both ends and the Euler–Lagrange theorem applies as stated. Here $F_u = -f$ and $F_{u'} = EA\,u'$, so

$$\big(EA\,u'\big)' + f = 0, \tag{10}$$

which is the equilibrium equation, $N' + f = 0$, with axial force $N := EA\,u'$. The differential equation together with the boundary conditions is a boundary value problem (BVP) derived from Π. As a shorthand, we say *the BVP has parent* Π. How Neumann and other non-Dirichlet boundary conditions are handled is discussed next.

Remark: The δ operator

We can treat δ as an operator that commutes with d/dx and with $\int$, so $\delta(u') = (\delta u)'$. We can therefore compute $\delta\Pi = \int_0^L \big(EA\,u'\,\delta u' - f\,\delta u\big)\,\mathrm{d}x$ directly, matching the theorem. This shorthand is useful for variational calculations throughout the course.

3. Essential and natural boundary conditions

The first-variation left a boundary term. We discuss what that term does when u is not prescribed.

Definition: Essential and natural boundary conditions

A boundary condition is essential if it is imposed on the admissible set, u is prescribed there and every variation satisfies $\eta = 0$ at that point. A boundary condition is natural if it is not imposed at all but emerges from variational stationarity at a boundary where u is free.

Proposition: Natural boundary condition

Let u make $J[u] = \int_0^L F(x, u, u')\,\mathrm{d}x$ stationary, with u prescribed at $x = 0$ and free at $x = L$. Then u satisfies the Euler–Lagrange equation on $(0, L)$ and,

$$\left.\frac{\partial F}{\partial u'}\right|_{x=L} = 0. \tag{11}$$

Proof. First restrict to variations with $\eta(L) = 0$. These recover the Euler–Lagrange equation, so the interior integral vanishes for all admissible η. For a general η, with $\eta(0) = 0$ but $\eta(L)$ arbitrary, the first-variation theorem then leaves $\delta J = F_{u'}(L)\,\eta(L) = 0\ \forall\,\eta(L)$, hence $F_{u'}(L) = 0$. □

Example: Bar with a free end

Take the running example with only $u(0) = 0$ prescribed. Here $F_{u'} = EA\,u'$, so the natural condition reads $N(L) = EA\,u'(L) = 0$, zero axial force at the free end. The functional was already compatible with a traction-free boundary.

Remark: Existence of a parent

Now suppose at $x = L$ we have a prescribed load $P \neq 0$, so the BC is $N(L) = EA\,u'(L) = P$. Is this boundary value problem (BVP) a child of Π? Not with $u(L)$ free, since stationarity forces the natural condition $N(L) = 0$. Forcing the data into the admissible set does not help either. Imposing $u'(L) = P/EA$ forces the variations to satisfy $\eta(0) = 0$ and $\eta'(L) = 0$ with $\eta(L)$ still free. By the richness remark the Euler–Lagrange equation still follows, and the surviving boundary term demands $EA\,u'(L)\,\eta(L) = P\,\eta(L) = 0$ for arbitrary $\eta(L)$, that is $P = 0$. Thus, unless the imposed BC coincides with the natural one, the BVP is not the child of Π. We show below that non-Dirichlet data belong in the functional, not in the space.

Example: Bar with an end load

To incorporate $N(L) = P \neq 0$ as the BC, consider the modified functional,

$$\Pi[u] = \int_0^L \Big(\tfrac{1}{2}\,EA\,(u')^2 - f\,u\Big)\,\mathrm{d}x \;-\; P\,u(L), \qquad u(0) = 0. \tag{12}$$

The first variation is

$$\delta\Pi = \int_0^L \big[-(EA\,u')' - f\big]\,\eta\;\mathrm{d}x \;+\; \big(EA\,u'(L) - P\big)\,\eta(L), \tag{13}$$

and stationarity gives $(EA\,u')' + f = 0$ in the interior and $N(L) = P$ at the boundary. Notice the load is not imposed on the variational space, but instead on the functional.

Theorem: Non-Dirichlet boundary conditions

Consider the variational form

$$J[u] = \int_0^L F(x, u, u')\;\mathrm{d}x + B\big(u(L)\big), \qquad u(0) = \bar{u}, \tag{14}$$

with B a smooth function of the end value. Stationarity gives the Euler–Lagrange equation together with the natural boundary condition,

$$\left.\frac{\partial F}{\partial u'}\right|_{x=L} + B'\big(u(L)\big) = 0. \tag{15}$$

Proof. The first variation of the functional, using $\eta(0) = 0$, yields

$$\delta J = \int_0^L \Big[F_u - \big(F_{u'}\big)'\Big]\,\eta\;\mathrm{d}x \;+\; \Big(F_{u'}(L) + B'\big(u(L)\big)\Big)\,\eta(L). \tag{16}$$

Applying the same two-step argument: variations with $\eta(L) = 0$ recover the Euler–Lagrange equation, and general variations with $\eta(L)$ being free force the boundary factor to vanish. $\square$

Example: Neumann and Robin boundary conditions

For $B = -P\,u(L)$, the natural condition is the Neumann boundary condition $F_{u'}(L) = P$. For $B = \frac{1}{2}\,k\,u(L)^2$, we get the homogeneous Robin boundary condition

$$F_{u'}(L) + k\,u(L) = 0, \tag{17}$$

which for the bar corresponds to an elastic support attached in *parallel*, $N(L) = -k\,u(L)$. For $B = \frac{1}{2}\,k\,u(L)^2 - P\,u(L)$, we get the nonhomogeneous Robin boundary condition

$$F_{u'}(L) + k\,u(L) = P, \tag{18}$$

which for the bar corresponds to an elastic support pulled by an end load, $N(L)+k\,u(L) = P$. Notice any Robin BC of the general form $c_1\,u'(L) + c_2\,u(L) = c_3$ with $c_1 \neq 0$ can be written in the stated form by dividing through c_1 and defining $k = EA\,c_2/c_1$ and $P = EA\,c_3/c_1$.

4. Functionals with higher derivatives

Theorem: Euler–Lagrange with second derivatives

For $J[w] = \int_0^L F(x, w, w', w'')\;\mathrm{d}x$, stationarity yields

$$\frac{\partial F}{\partial w} - \frac{d}{dx}\frac{\partial F}{\partial w'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial w''} = 0 \quad \text{on } (0, L), \tag{19}$$

and two independent alternative BCs at each end,

$$w = \bar{w} \quad \text{or} \quad F_{w'} - \big(F_{w''}\big)' = 0, \tag{20}$$

and

$$w' = \bar{w}' \quad \text{or} \quad F_{w''} = 0. \tag{21}$$

Proof. Two integrations by parts give

$$\delta J = \int_0^L \Big[F_w - (F_{w'})' + (F_{w''})''\Big]\eta\;\mathrm{d}x + \Big[\big(F_{w'} - (F_{w''})'\big)\,\eta\Big]_0^L + \Big[F_{w''}\,\eta'\Big]_0^L, \tag{22}$$

the richness remark recovers the interior equation. The BCs are recovered from the fact that η and η' are free at any end where w and w' are not prescribed, and vanish where they are prescribed. $\square$

Example: Euler–Bernoulli beam

Consider a beam $x \in (0, L)$ subject to transverse load $q(x)$ per unit length, causing it to deflect by $w(x)$. The Euler–Bernoulli hypothesis is that cross-sections remain planar and perpendicular to the *deformed* axis. This means each cross-section rotates with local slope w', so a fiber at height y above the axis is displaced axially by $-y\,w'(x)$, with axial strain

$$\varepsilon(x, y) = \frac{\partial}{\partial x}\big(-y\,w'\big) = -y\,w''. \tag{23}$$

The strain energy per unit length is therefore

$$\int_A \tfrac{1}{2}\,E\,\big(y\,w''\big)^2\,\mathrm{d}A = \tfrac{1}{2}\,E\,I\,(w'')^2, \qquad I := \int_A y^2\,\mathrm{d}A, \tag{24}$$

with I the second moment of area of the cross-section A. Defining the functional,

$$\Pi[w] = \int_0^L \Big(\tfrac{1}{2}\,EI\,(w'')^2 - q\,w\Big)\,\mathrm{d}x. \tag{25}$$

The first variation, after two integrations by parts, is

$$\delta\Pi = \int_0^L \Big[(EI\,w'')'' - q\Big]\,\eta\,\mathrm{d}x \;+\; \Big[EI\,w''\,\eta'\Big]_0^L \;-\; \Big[(EI\,w'')'\,\eta\Big]_0^L, \tag{26}$$

so the interior equation is $(EI\,w'')'' = q$. The factor multiplying η' is the bending moment,

$$M := EI\,w'' = -\int_A y\,\sigma\,\mathrm{d}A, \qquad \sigma = E\,\varepsilon, \tag{27}$$

the resultant of the axial stresses. The factor multiplying η is the shear force,

$$V := -(EI\,w'')' = -M', \tag{28}$$

the resultant of the shear stresses on the cross-section. This is because moment balance of a dx–slice at x consists of pure couples $M(x)$ and $M(x + dx)$ due to axial stresses (which net to zero) and a shear torque at $x + dx$. Hence,

$$dM + V\,dx = 0 \quad \Rightarrow \quad V = -M' \tag{29}$$

Compatible BCs now are: (1) At a clamped end, cross-section can neither translate nor rotate, thus $w = 0$ and $w' = 0$; (2) At a simply supported end $w = 0$ while rotation is free, so $M = 0$ is natural; (3) At a free end η and η' are both free, so $V = M = 0$. For nonhomogeneous BCs, subtracting $\bar{P}\,w(L) + \bar{M}\,w'(L)$ from Π turns boundary terms at $x = L$

$$\big(V(L) - \bar{P}\big)\,\eta(L) + \big(M(L) - \bar{M}\big)\,\eta'(L), \tag{30}$$

so stationarity yields $V(L) = \bar{P}$ and $M(L) = \bar{M}$, the beam version of Non-Dirichlet theorem.

5. From stationarity to minimum, and the weak form

Definition: Second variation

The second variation of a functional J is defined as

$$\delta^2 J := \frac{d^2}{d\epsilon^2}\, J[u+\epsilon\eta]\,\Big|_{\epsilon=0}, \tag{31}$$

so that the perturbed functional has the Taylor expansion

$$J[u+\epsilon\eta] = J[u] + \epsilon\,\delta J + \frac{\epsilon^2}{2}\,\delta^2 J + O(\epsilon^3). \tag{32}$$

Remark: Second variation of the integral functional

For $J[u] = \int_0^L F(x,u,u')\,\mathrm{d}x$, the second variation definition yields

$$\delta^2 J = \int_0^L \Big(F_{uu}\,\eta^2 + 2\,F_{uu'}\,\eta\,\eta' + F_{u'u'}\,(\eta')^2\Big)\,\mathrm{d}x, \tag{33}$$

where $F_{uu'} = F_{u'u}$ was used to combine the mixed partial derivatives. The above result can also be derived using the δ shorthand. Write the first variation as

$$\delta J = \int_0^L \big(F_u\,\eta + F_{u'}\,\eta'\big)\,\mathrm{d}x, \tag{34}$$

and apply δ once more with the product rule, $\delta F_u = F_{uu}\,\eta + F_{uu'}\,\eta'$ and $\delta F_{u'} = F_{u'u}\,\eta + F_{u'u'}\,\eta'$, while $\delta\eta = \delta\eta' = 0$ because the perturbation direction η is held fixed. Equivalently $\delta(\delta u) = \delta^2 u = 0$, which follows directly from the definition, $\delta^2 u = d^2/d\epsilon^2\,\big(u+\epsilon\eta\big)\big|_{\epsilon=0} = 0$.

Example: Stationarity implies minimum for the bar

Let u be the stationary point of the loaded-bar functional

$$\Pi[u] = \int_0^L \Big(\tfrac{1}{2}\,EA\,(u')^2 - f\,u\Big)\,\mathrm{d}x - P\,u(L), \qquad u(0) = 0, \tag{35}$$

over the admissible set. Its first and second variations are

$$\delta\Pi = \int_0^L \big(EA\,u'\,\eta' - f\,\eta\big)\,\mathrm{d}x - P\,\eta(L), \qquad \delta^2\Pi = \int_0^L EA\,(\eta')^2\,\mathrm{d}x, \tag{36}$$

the latter from the remark with $F_{u'u'} = EA$ as the only nonzero second derivative. Every higher variation vanishes, since applying δ to $\delta^2\Pi$ gives $\int_0^L 2\,EA\,\eta'\,\delta\eta'\,\mathrm{d}x = 0$ by $\delta\eta' = 0$. Stationarity gives $\delta\Pi = 0$, so the Taylor expansion terminates at second order,

$$\Pi[u+\epsilon\eta] = \Pi[u] + \frac{\epsilon^2}{2}\int_0^L EA\,(\eta')^2\,\mathrm{d}x, \tag{37}$$

The second-order term is strictly positive unless $\eta' \equiv 0$, which together with $\eta(0) = 0$ forces $\eta \equiv 0$, the trivial perturbation. Hence $\Pi[u+\epsilon\eta] > \Pi[u]$ for every nonzero admissible perturbation, and u is the unique global minimizer. The beam is identical with $\delta^2\Pi = \int_0^L EI\,(\eta'')^2\,\mathrm{d}x$. Thus equilibrium, stationarity, and minimum total potential energy coincide.

Definition: The weak form

Stationarity of $\Pi[u]$ can be written as

$$\int_0^L EA\,u'\,\eta'\,\mathrm{d}x = \int_0^L f\,\eta\,\mathrm{d}x + P\,\eta(L) \qquad \forall \text{ admissible } \eta, \tag{38}$$

a statement requiring one derivative of u instead of two in the strong form.

This is the form of the governing equation that a computer discretizes in the finite element method. This is also intimately related to the principle of virtual work, which we will discuss next.