

Mechanics of Deformable Porous Media

Fundamentals, Slot 1: Continuum Mechanics Essentials

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This slot is a brisk review that fixes language, sign conventions, and notation. Everything here is used repeatedly later in the course.

1. Tensors and index notation

Definition: Index notation

A repeated index is summed over 1, 2, 3. Examples:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i, \quad (\boldsymbol{\sigma} \mathbf{n})_i = \sigma_{ij} n_j, \quad (\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k. \quad (1)$$

Kronecker delta and permutation symbol:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad \delta_{ij} a_j = a_i, \quad \epsilon_{ijk} = \begin{cases} +1 & ijk \text{ even permutation of } 123 \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{repeated index.} \end{cases} \quad (2)$$

Differential operators:

$$(\nabla \mathbf{u})_{ij} = u_{i,j}, \quad (\nabla \cdot \boldsymbol{\sigma})_i = \sigma_{ij,j}, \quad \nabla \cdot \mathbf{u} = u_{i,i}. \quad (3)$$

The following one-line lemma is used repeatedly, first in the proof of stress symmetry below.

Lemma: Antisymmetric-symmetric contraction

If $A_{jk} = -A_{kj}$ and $S_{jk} = S_{kj}$, then $A_{jk} S_{jk} = 0$.

Proof. Relabel dummy indices: $A_{jk} S_{jk} = A_{kj} S_{kj} = -A_{jk} S_{jk}$, hence sum equals its own negative. $\square$

2. Traction and stress

Definition: Traction vector

Cut the body at $\mathbf{x}$ along a surface with unit normal $\mathbf{n}$. The traction is the force per unit area transmitted across the cut,

$$\mathbf{t}(\mathbf{n}, \mathbf{x}) = \lim_{\Delta A \rightarrow 0} \frac{\Delta \mathbf{f}}{\Delta A}. \quad (4)$$

Theorem: Cauchy's stress theorem

The dependence of $\mathbf{t}$ on $\mathbf{n}$ is linear. There exists a second-order tensor field $\boldsymbol{\sigma}(\mathbf{x})$ such that

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n}, \quad t_i = \sigma_{ij} n_j. \quad (5)$$

Proof sketch (Cauchy tetrahedron). Take a small tetrahedron at $\mathbf{x}$ with three faces on the coordinate planes and an oblique face of area ΔA and normal $\mathbf{n}$. The coordinate faces have areas $n_i \Delta A$. Newton's second law contains surface terms of order $\Delta A \sim h^2$ and volume terms (body force, inertia) of order h^3 . Divide by ΔA and let $h \rightarrow 0$. The volume terms vanish and what remains is $t_i(\mathbf{n}) = t_i(\mathbf{e}_j) n_j =: \sigma_{ij} n_j$. Linearity in $\mathbf{n}$ is therefore a consequence of momentum balance. $\square$

3. Balance laws

Theorem: Local balance of linear momentum

For every subbody $\omega \subseteq \Omega$, global balance $\int_{\partial\omega} \mathbf{t} \, dA + \int_{\omega} \mathbf{b} \, dV = \int_{\omega} \rho \ddot{\mathbf{u}} \, dV$ together with Cauchy's theorem and the divergence theorem implies the local form

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}} \quad \text{in } \Omega \quad (\text{statics: } \nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}), \quad (6)$$

with boundary conditions $\boldsymbol{\sigma} \mathbf{n} = \bar{\mathbf{t}}$ on Γ_t and $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_u .

Proof. Insert $\mathbf{t} = \boldsymbol{\sigma} \mathbf{n}$ into the surface integral, convert it with the divergence theorem, and collect everything into one volume integral over ω . Since ω is arbitrary, the integrand vanishes. $\square$

Theorem: Symmetry of the stress tensor

Balance of angular momentum, in the absence of body couples, implies

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top, \quad \sigma_{ij} = \sigma_{ji}. \quad (7)$$

Proof. Angular momentum about the origin for any subbody ω :

$$\frac{d}{dt} \int_{\omega} \rho \epsilon_{ijk} x_j \dot{u}_k \, dV = \int_{\partial\omega} \epsilon_{ijk} x_j t_k \, dA + \int_{\omega} \epsilon_{ijk} x_j b_k \, dV. \quad (8)$$

Left side: mass conservation lets the time derivative pass through $\rho \, dV$, and since $\mathbf{x} = \mathbf{X} + \mathbf{u}$ with $\mathbf{X}$ fixed we have $\dot{x}_j = \dot{u}_j$. The product rule gives

$$\int_{\omega} \rho \epsilon_{ijk} (\dot{u}_j \dot{u}_k + x_j \ddot{u}_k) \, dV = \int_{\omega} \rho \epsilon_{ijk} x_j \ddot{u}_k \, dV, \quad (9)$$

because $\epsilon_{ijk} \dot{u}_j \dot{u}_k = (\dot{\mathbf{u}} \times \dot{\mathbf{u}})_i = 0$ by the Lemma of Section 1 (ϵ_{ijk} is antisymmetric in jk , the dyad $\dot{u}_j \dot{u}_k$ is symmetric). Surface term, using $t_k = \sigma_{kl} n_l$, the divergence theorem, and $x_{j,l} = \delta_{jl}$:

$$\int_{\partial\omega} \epsilon_{ijk} x_j \sigma_{kl} n_l \, dA = \int_{\omega} (\epsilon_{ijk} x_j \sigma_{kl})_{,l} \, dV = \int_{\omega} \epsilon_{ijk} \sigma_{kj} \, dV + \int_{\omega} \epsilon_{ijk} x_j \sigma_{kl,l} \, dV. \quad (10)$$

Collecting all terms,

$$\int_{\omega} \epsilon_{ijk} x_j \underbrace{(\sigma_{kl,l} + b_k - \rho \ddot{u}_k)}_{=0 \text{ by linear momentum}} \, dV + \int_{\omega} \epsilon_{ijk} \sigma_{kj} \, dV = 0. \quad (11)$$

Localization gives $\epsilon_{ijk} \sigma_{kj} = 0$ for each i . For $i = 1, 2, 3$ yields $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{12} = \sigma_{21}$. $\square$

Physical Insight

Angular momentum is not a free lunch. It costs exactly the symmetry of $\boldsymbol{\sigma}$. In media that carry distributed body couples (micropolar theories) the stress tensor is not symmetric.

Definition: Principal stresses

Since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$, it has real eigenvalues σ_a and an orthonormal set of eigenvectors $\mathbf{n}_a$, giving the spectral representation

$$\boldsymbol{\sigma} = \sum_{a=1}^3 \sigma_a \mathbf{n}_a \otimes \mathbf{n}_a. \quad (12)$$

On principal planes, traction is purely normal. We see this form later, when the tension–compression split of phase-field fracture is built from the eigenpairs of the strain tensor.

4. Strain

Definition: Small-strain tensor

For a displacement field $\mathbf{u}(\mathbf{x}, t)$, deformation is measured by the symmetric gradient

$$\boldsymbol{\varepsilon} = \text{sym } \nabla \mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^\top), \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \quad (13)$$

The full gradient splits as $\nabla \mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$, where the skew part $\boldsymbol{\omega}$ is the infinitesimal rotation. Rotations store no energy, which is why only $\boldsymbol{\varepsilon}$ enters constitutive laws. The theory is valid for $|\nabla \mathbf{u}| \ll 1$. Finite strains come later in the course.

Definition: Volumetric–deviatoric split

$$\varepsilon_v = \text{tr } \boldsymbol{\varepsilon} = \nabla \cdot \mathbf{u}, \quad \mathbf{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\varepsilon_v \mathbf{1}, \quad \sigma_m = \frac{1}{3} \text{tr } \boldsymbol{\sigma}, \quad \mathbf{s} = \boldsymbol{\sigma} - \sigma_m \mathbf{1}, \quad (14)$$

with $\text{tr } \mathbf{e} = \text{tr } \mathbf{s} = 0$. To first order, ε_v is the relative volume change.

Remark: Sign conventions and effective stress

Stress and strain are tension-positive. Pore pressure p is compression-positive. In a fluid-saturated medium the momentum balance sees only the *total* stress $\boldsymbol{\sigma}$, while the constitutive laws of the solid skeleton (elasticity below, yield surfaces later) act on *effective* stress $\boldsymbol{\sigma}'$,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - b p \mathbf{1} \quad (15)$$

with Biot coefficient $b \leq 1$ ($b = 1$ recovers Terzaghi). Derivation comes later in the course.

Example. A sealed sample compressed hydrostatically to $\sigma_m = -100$ with pore pressure $p = 40$ has effective mean stress $\sigma'_m = -60$. Of the 100, the skeleton carries 60 and the pore fluid carries 40.

Remark: Compatibility

The operator $\text{sym } \nabla$ maps 3 displacement components into 6 strain components, so the inverse question is over-determined. Given a symmetric field $\boldsymbol{\varepsilon}(\mathbf{x})$, a single-valued displacement with $\text{sym } \nabla \mathbf{u} = \boldsymbol{\varepsilon}$ exists only if $\boldsymbol{\varepsilon}$ satisfies the Saint-Venant compatibility conditions

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}, \quad (16)$$

equivalently $\nabla \times (\nabla \times \boldsymbol{\varepsilon}) = \mathbf{0}$, six independent scalar equations. In 2D, this reduces to one:

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy}. \quad (17)$$

They are always necessary, and sufficient on simply connected domains. On a domain with holes, incompatibility can hide in the topology, which is how dislocations are modeled. The practical point is a division of labor between two energy principles. In displacement-based formulations $\mathbf{u}$ is primal, compatibility holds by construction, and the variational principle delivers equilibrium. In stress-based formulations the roles reverse, equilibrium is assumed and compatibility is what the principle delivers. We will see both later in energy methods.

5. Linear elasticity

Definition: Strain energy density

A material is hyperelastic if there exists a potential $\psi(\boldsymbol{\varepsilon})$ such that

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}. \quad (18)$$

In the linear case $\psi = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$, that is $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$.

Proposition: Minor symmetries of the stiffness tensor

The minor symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk} \quad (19)$$

are equivalent to the symmetry of the stress and strain tensors.

Proof. Since $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\top$, relabeling dummy indices gives $C_{ijkl} \varepsilon_{kl} = C_{ijlk} \varepsilon_{lk} = C_{ijlk} \varepsilon_{kl}$, so only the part of $\mathbb{C}$ symmetric in (kl) ever acts on a strain and nothing is lost by taking $C_{ijkl} = C_{ijlk}$. Since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$ (angular mom), $\sigma_{ij} = \sigma_{ji}$ gives $C_{ijkl} \varepsilon_{kl} = C_{jikl} \varepsilon_{kl}$ for every $\boldsymbol{\varepsilon}$, hence $C_{ijkl} = C_{jikl}$. $\square$

Proposition: Major symmetry $\Leftrightarrow$ existence of ψ

Suppose a linear constitutive law is *given*,

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \mathbb{C} : \boldsymbol{\varepsilon}. \quad (20)$$

We say this law *admits a potential* if there exists a scalar function $\psi(\boldsymbol{\varepsilon})$ such that

$$\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \quad \forall \boldsymbol{\varepsilon}. \quad (21)$$

A potential exists if and only if $C_{ijkl} = C_{klij}$.

Proof. ($\Rightarrow$) Assume $C_{ijkl} = C_{klij}$ and construct $\psi := \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}$. Then

$$\frac{\partial \psi}{\partial \varepsilon_{ij}} = \frac{1}{2} (C_{ijkl} + C_{klij}) \varepsilon_{kl} = C_{ijkl} \varepsilon_{kl} = \sigma_{ij}(\boldsymbol{\varepsilon}), \quad (22)$$

where the middle equality uses the major symmetry, so ψ is a potential for the law.

($\Leftarrow$) If a potential ψ exists, then

$$C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 \psi}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}}, \quad (23)$$

and mixed partial derivatives commute, so $C_{ijkl} = C_{klij}$. $\square$

Physical Insight

A linear law with major symmetry is called hyperelastic (or Green elastic). Without it the law is still mathematically well-defined (Cauchy elastic) but carries no energy function: the work $\int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}$ becomes path dependent, and a closed strain cycle outputs net work, an elastic perpetuum mobile. Energy conservation, not convenience, forces the major symmetry.

Proposition: Counting elastic constants

The stiffness tensor C_{ijkl} has at most 21 independent material constants. Material symmetry reduces this further: orthotropic 9, transversely isotropic 5, cubic 3, isotropic 2.

Proof. Without symmetry, C_{ijkl} has 3^4 constants. A symmetric index pair in dimension n carries

$$\frac{n^2 - n}{2} + n = \frac{n(n+1)}{2} \quad (24)$$

independent values (the diagonal plus half of the off-diagonal), which is 6 for $n = 3$. The minor symmetries make each of the pairs (ij) and (kl) a symmetric pair, leaving a 6×6 array of 36 constants, the maximum any linear elastic material can have. The major symmetry states that this array is itself symmetric, so the same formula with $n = 6$ gives

$$\frac{36 - 6}{2} + 6 = 21. \quad \square \quad (25)$$

Theorem: Isotropic linear elasticity

If the material is isotropic, the stiffness tensor

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (26)$$

carries exactly two independent constants λ and μ (the Lamé coefficients). Thus,

$$\boldsymbol{\sigma} = \lambda \varepsilon_v \mathbf{1} + 2\mu \boldsymbol{\varepsilon}, \quad \psi = \frac{1}{2} \lambda \varepsilon_v^2 + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}. \quad (27)$$

Proof. A classical result of invariant theory (Weyl) states that every isotropic tensor of even order is a linear combination of products of Kronecker deltas. For fourth order,

$$C_{ijkl} = a \delta_{ij} \delta_{kl} + b \delta_{ik} \delta_{jl} + c \delta_{il} \delta_{jk}. \quad (28)$$

Swapping $i \leftrightarrow j$ exchanges b and c , so minor symmetry forces $b = c$. Set $\lambda := a$ and $\mu := b$. Major symmetry then holds automatically, and contracting with $\boldsymbol{\varepsilon}$ yields the stated forms for $\boldsymbol{\sigma}$ and ψ . $\square$

Remark: Alternate moduli and forms

Isotropic materials are uniquely determined by two moduli, and any two determine the rest:

	from (λ, μ)	from (K, μ)	from (E, ν)
λ	λ	$K - \frac{2}{3}\mu$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
μ	μ	μ	$\frac{E}{2(1+\nu)}$
K	$\lambda + \frac{2}{3}\mu$	K	$\frac{E}{3(1-2\nu)}$
E	$\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$	$\frac{9K\mu}{3K + \mu}$	E
ν	$\frac{\lambda}{2(\lambda + \mu)}$	$\frac{3K - 2\mu}{2(3K + \mu)}$	ν

Hooke's law and energy decouple into volumetric and deviatoric terms using K and μ ,

$$\sigma_m = K \varepsilon_v, \quad \mathbf{s} = 2\mu \mathbf{e}, \quad \psi = \frac{1}{2} K \varepsilon_v^2 + \mu \mathbf{e} : \mathbf{e}. \quad (29)$$

Note that $\psi > 0$ for every $\varepsilon \neq \mathbf{0}$ if and only if $K > 0$ and $\mu > 0$, equivalently $E > 0$ and $-1 < \nu < \frac{1}{2}$. We will use (29) in the lecture on plasticity.

Remark: Symmetry and potentials

A recurring theme throughout this course is that a potential exists if and only if the underlying operator is symmetric. (i) Algebraic: the linear map $\varepsilon \mapsto \mathbb{C} : \varepsilon$ on the six-dimensional space of symmetric tensors. Major symmetry says this map is self-adjoint with respect to the inner product $\mathbf{A} : \mathbf{B}$, hence ψ exists. (ii) Integral: Betti's reciprocity theorem is a consequence of the existence of the strain-energy functional. (iii) Differential: a system of PDEs with boundary conditions descends from a functional if and only if its linearization is formally self-adjoint. (iv) Kinetic: Onsager reciprocity is the condition for a dissipation potential to exist. We will discuss (ii) and (iv) later in the course.