Mechanics of Deformable Porous Media

Fundamentals, Slot 1: Continuum Mechanics Essentials

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This slot is a brisk review that fixes language, sign conventions, and notation. Everything here is used repeatedly later in the course.

1. Tensors and index notation

Definition: Index notation

A repeated index is summed over $1, 2, 3$. Examples:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_i b_i, \qquad (\boldsymbol{\sigma}\,\boldsymbol{n})_i = \sigma_{ij} n_j, \qquad (\boldsymbol{a}\times\boldsymbol{b})_i = \epsilon_{ijk} a_j b_k. \tag{1}$$

Kronecker delta and permutation symbol:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \qquad \delta_{ij} a_j = a_i, \qquad \epsilon_{ijk} = \begin{cases} +1 & ijk \text{ even permutation of } 123 \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{repeated index.} \end{cases} \tag{2}$$

Differential operators:

$$(\nabla \boldsymbol{u})_{ij} = u_{i,j}, \qquad (\nabla\cdot\boldsymbol{\sigma})_i = \sigma_{ij,j}, \qquad \nabla\cdot\boldsymbol{u} = u_{i,i}. \tag{3}$$

The following one-line lemma is used repeatedly, first in the proof of stress symmetry below.

Lemma: Antisymmetric–symmetric contraction

If $A_{jk} = -A_{kj}$ and $S_{jk} = S_{kj}$, then $A_{jk} S_{jk} = 0$.

Proof. Relabel dummy indices: $A_{jk} S_{jk} = A_{kj} S_{kj} = -A_{jk} S_{jk}$, hence sum equals its own negative. □

2. Traction and stress

Definition: Traction vector

Cut the body at $\boldsymbol{x}$ along a surface with unit normal $\boldsymbol{n}$. The traction is the force per unit area transmitted across the cut,

$$\boldsymbol{t}(\boldsymbol{n}, \boldsymbol{x}) = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{f}}{\Delta A}. \tag{4}$$

Theorem: Cauchy's stress theorem

The dependence of $\boldsymbol{t}$ on $\boldsymbol{n}$ is linear. There exists a second-order tensor field $\boldsymbol{\sigma}(\boldsymbol{x})$ such that

$$\boldsymbol{t} = \boldsymbol{\sigma}\,\boldsymbol{n}, \qquad t_i = \sigma_{ij} n_j. \tag{5}$$

Proof sketch (Cauchy tetrahedron). Take a small tetrahedron at $\boldsymbol{x}$ with three faces on the coordinate planes and an oblique face of area ΔA and normal $\boldsymbol{n}$. The coordinate faces have areas $n_i\,\Delta A$. Newton's second law contains surface terms of order $\Delta A \sim h^2$ and volume terms (body force, inertia) of order h^3. Divide by ΔA and let $h \to 0$. The volume terms vanish and what remains is $t_i(\boldsymbol{n}) = t_i(\boldsymbol{e}_j)\, n_j =: \sigma_{ij} n_j$. Linearity in $\boldsymbol{n}$ is therefore a consequence of momentum balance. □

3. Balance laws

Theorem: Local balance of linear momentum

For every subbody $\omega \subseteq \Omega$, global balance $\int_{\partial\omega} \boldsymbol{t}\,\mathrm{d}A + \int_{\omega} \boldsymbol{b}\,\mathrm{d}V = \frac{d}{dt}\int_{\omega} \rho\dot{\boldsymbol{u}}\,\mathrm{d}V$ together with Cauchy's theorem and the divergence theorem implies the local form:

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \rho\,\ddot{\boldsymbol{u}} \;\text{ in } \Omega \qquad (\text{statics: } \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}), \tag{6}$$

The local form can be solved with boundary conditions $\boldsymbol{\sigma}\,\boldsymbol{n} = \bar{\boldsymbol{t}}$ on Γ_t and $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_u.

Proof. Insert $\boldsymbol{t} = \boldsymbol{\sigma}\,\boldsymbol{n}$ into the surface integral, convert it with the divergence theorem, move d/dt into integrand to get $\rho\ddot{\boldsymbol{u}}$ as ρdV is conserved, then collect everything into one integral over ω. Since ω is arbitrary, the integrand vanishes and the local form is recovered. □

Theorem: Symmetry of the stress tensor

Balance of angular momentum, in the absence of body couples, implies

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\top}, \qquad \sigma_{ij} = \sigma_{ji}. \tag{7}$$

Proof. Angular momentum about the origin for any subbody ω:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\omega \rho\,\epsilon_{ijk}\,x_j\dot{u}_k\,\mathrm{d}V = \int_{\partial\omega}\epsilon_{ijk}\,x_j t_k\,\mathrm{d}A + \int_\omega \epsilon_{ijk}\,x_j b_k\,\mathrm{d}V. \tag{8}$$

Left side: mass conservation lets the time derivative pass through $\rho\,\mathrm{d}V$, and since $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}$ with $\boldsymbol{X}$ fixed we have $\dot{x}_j = \dot{u}_j$. The product rule gives

$$\int_\omega \rho\,\epsilon_{ijk}\big(\dot{u}_j\dot{u}_k + x_j\ddot{u}_k\big)\,\mathrm{d}V = \int_\omega \rho\,\epsilon_{ijk}\,x_j\ddot{u}_k\,\mathrm{d}V, \tag{9}$$

because $\epsilon_{ijk}\dot{u}_j\dot{u}_k = (\dot{\boldsymbol{u}}\times\dot{\boldsymbol{u}})_i = 0$ by the Lemma of Section 1 (ϵ_{ijk} is antisymmetric in jk, the dyad $\dot{u}_j\dot{u}_k$ is symmetric). Surface term, using $t_k = \sigma_{kl}n_l$, the divergence theorem, and $x_{j,l} = \delta_{jl}$:

$$\int_{\partial\omega}\epsilon_{ijk}\,x_j\sigma_{kl}n_l\,\mathrm{d}A = \int_\omega \big(\epsilon_{ijk}\,x_j\sigma_{kl}\big)_{,l}\,\mathrm{d}V = \int_\omega \epsilon_{ijk}\,\sigma_{kj}\,\mathrm{d}V + \int_\omega \epsilon_{ijk}\,x_j\,\sigma_{kl,l}\,\mathrm{d}V. \tag{10}$$

Collecting all terms,

$$\int_\omega \epsilon_{ijk}\,x_j\,\underbrace{\big(\sigma_{kl,l} + b_k - \rho\ddot{u}_k\big)}_{=0\text{ by linear momentum}}\,\mathrm{d}V + \int_\omega \epsilon_{ijk}\,\sigma_{kj}\,\mathrm{d}V = 0. \tag{11}$$

Localization gives $\epsilon_{ijk}\sigma_{kj} = 0$ for each i. For $i = 1,2,3$ yields $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{12} = \sigma_{21}$. □

Physical Insight

Angular momentum is not a free lunch. It costs exactly the symmetry of $\boldsymbol{\sigma}$. In media that carry distributed body couples (micropolar theories) the stress tensor is not symmetric.

Definition: Principal stresses

Since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$, it has real eigenvalues σ_a and an orthonormal set of eigenvectors $\boldsymbol{n}_a$, giving the spectral representation

$$\boldsymbol{\sigma} = \sum_{a=1}^{3}\sigma_a\,\boldsymbol{n}_a\otimes\boldsymbol{n}_a. \tag{12}$$

On principal planes, traction is purely normal. We see this form later, when the tension–compression split of phase-field fracture is built from the eigenpairs of the strain tensor.

4. Strain

Definition: Small-strain tensor

For a displacement field $\boldsymbol{u}(\boldsymbol{x},t)$, deformation is measured by the symmetric gradient

$$\boldsymbol{\varepsilon} = \operatorname{sym}\nabla\boldsymbol{u} = \tfrac{1}{2}\big(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top\big), \qquad \varepsilon_{ij} = \tfrac{1}{2}\big(u_{i,j} + u_{j,i}\big). \tag{13}$$

The full gradient splits as $\nabla \boldsymbol{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$, where the skew part $\boldsymbol{\omega}$ is the infinitesimal rotation. Rotations store no energy, which is why only $\boldsymbol{\varepsilon}$ enters constitutive laws. The theory is valid for $|\nabla \boldsymbol{u}| \ll 1$. Finite strains come later in the course.

Definition: Volumetric–deviatoric split

$$\varepsilon_v = \operatorname{tr}\boldsymbol{\varepsilon} = \nabla \cdot \boldsymbol{u}, \qquad \boldsymbol{e} = \boldsymbol{\varepsilon} - \tfrac{1}{3}\varepsilon_v \mathbf{1}, \qquad \sigma_m = \tfrac{1}{3}\operatorname{tr}\boldsymbol{\sigma}, \qquad \boldsymbol{s} = \boldsymbol{\sigma} - \sigma_m \mathbf{1}, \tag{14}$$

with $\operatorname{tr}\boldsymbol{e} = \operatorname{tr}\boldsymbol{s} = 0$. To first order, ε_v is the relative volume change.

Remark: Sign conventions and effective stress

Stress and strain are tension-positive. Pore pressure p is compression-positive. In a fluid-saturated medium the momentum balance sees only the *total* stress $\boldsymbol{\sigma}$, while the constitutive laws of the solid skeleton (elasticity below, yield surfaces later) act on *effective* stress $\boldsymbol{\sigma}'$,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - b\,p\,\mathbf{1} \tag{15}$$

with Biot coefficient $b \le 1$ ($b = 1$ recovers Terzaghi). Derivation comes later in the course.

Example. A sealed sample compressed hydrostatically to $\sigma_m = -100$ with pore pressure $p = 40$ has effective mean stress $\sigma_m' = -60$. Of the 100, the skeleton carries 60 and the pore fluid carries 40.

Remark: Compatibility

The operator $\operatorname{sym}\nabla$ maps 3 displacement components into 6 strain components, so the inverse question is over-determined. Given a symmetric field $\boldsymbol{\varepsilon}(\boldsymbol{x})$, a single-valued displacement with $\operatorname{sym}\nabla\boldsymbol{u} = \boldsymbol{\varepsilon}$ exists only if $\boldsymbol{\varepsilon}$ satisfies the Saint-Venant compatibility conditions

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}, \tag{16}$$

equivalently $\nabla \times (\nabla \times \boldsymbol{\varepsilon}) = \mathbf{0}$, six independent scalar equations. In 2D, this reduces to one:

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\,\varepsilon_{xy,xy}. \tag{17}$$

They are always necessary, and sufficient on simply connected domains. On a domain with holes, incompatibility can hide in the topology, which is how dislocations are modeled. The practical point is a division of labor between two energy principles. In displacement-based formulations $\boldsymbol{u}$ is primal, compatibility holds by construction, and the variational principle delivers equilibrium. In stress-based formulations the roles reverse, equilibrium is assumed and compatibility is what the principle delivers. We will see both later in energy methods.

5. Linear elasticity

Definition: Strain energy density

A material is hyperelastic if there exists a potential $\psi(\boldsymbol{\varepsilon})$ such that

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}. \tag{18}$$

In the linear case $\psi = \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$, that is $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$.

Proposition: Minor symmetries of the stiffness tensor

The minor symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk} \tag{19}$$

are equivalent to the symmetry of the stress and strain tensors.

Proof. Since $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\top$, relabeling dummy indices gives $C_{ijkl}\,\varepsilon_{kl} = C_{ijlk}\,\varepsilon_{lk} = C_{ijlk}\,\varepsilon_{kl}$, so only the part of $\mathbb{C}$ symmetric in (kl) ever acts on a strain and nothing is lost by taking $C_{ijkl} = C_{ijlk}$. Since $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$ (angular mom), $\sigma_{ij} = \sigma_{ji}$ gives $C_{ijkl}\,\varepsilon_{kl} = C_{jikl}\,\varepsilon_{kl}$ for every $\boldsymbol{\varepsilon}$, hence $C_{ijkl} = C_{jikl}$. □

Proposition: Major symmetry ⇔ existence of ψ

Suppose a linear constitutive law is *given*,

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \mathbb{C} : \boldsymbol{\varepsilon}. \tag{20}$$

We say this law *admits a potential* if there exists a scalar function $\psi(\boldsymbol{\varepsilon})$ such that

$$\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \quad \forall\,\boldsymbol{\varepsilon}. \tag{21}$$

A potential exists if and only if $C_{ijkl} = C_{klij}$.

Proof. (⇒) Assume $C_{ijkl} = C_{klij}$ and construct $\psi := \frac{1}{2}\,\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}$. Then

$$\frac{\partial \psi}{\partial \varepsilon_{ij}} = \tfrac{1}{2}\big(C_{ijkl} + C_{klij}\big)\varepsilon_{kl} = C_{ijkl}\,\varepsilon_{kl} = \sigma_{ij}(\boldsymbol{\varepsilon}), \tag{22}$$

where the middle equality uses the major symmetry, so ψ is a potential for the law.

(⇐) If a potential ψ exists, then

$$C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 \psi}{\partial \varepsilon_{kl}\,\partial \varepsilon_{ij}}, \tag{23}$$

and mixed partial derivatives commute, so $C_{ijkl} = C_{klij}$. □

Physical Insight

A linear law with major symmetry is thus hyperelastic (or Green elastic). Without it the law is still mathematically well-defined (Cauchy elastic) but carries no energy function: the work $\int \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}$ becomes path dependent, and a closed strain cycle outputs net work, an elastic perpetuum mobile. Energy conservation, not convenience, forces the major symmetry.

Proposition: Counting elastic constants

The stiffness tensor C_{ijkl} has at most 21 independent material constants. Material symmetry reduces this further: orthotropic 9, transversely isotropic 5, cubic 3, isotropic 2.

Proof. Without symmetry, C_{ijkl} has 3^4 constants. A symmetric index pair in dimension n carries

$$\frac{n^2 - n}{2} + n = \frac{n(n+1)}{2} \tag{24}$$

independent values (the diagonal plus half of the off-diagonal), which is 6 for $n = 3$. The minor symmetries make each of the pairs (ij) and (kl) a symmetric pair, leaving a 6×6 array of 36 constants, the maximum any linear elastic material can have. The major symmetry states that this array is itself symmetric, so the same formula with $n = 6$ gives

$$\frac{36 - 6}{2} + 6 = 21. \qquad \square \tag{25}$$

Theorem: Isotropic linear elasticity

If the material is isotropic, the stiffness tensor

$$C_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big), \tag{26}$$

carries exactly two independent constants λ and μ (the Lamé coefficients). Thus,

$$\boldsymbol{\sigma} = \lambda\, \varepsilon_v\, \mathbf{1} + 2\mu\, \boldsymbol{\varepsilon}, \qquad \psi = \tfrac{1}{2}\, \lambda\, \varepsilon_v^2 + \mu\, \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}. \tag{27}$$

Proof. A classical result of invariant theory (Weyl) states that every isotropic tensor of even order is a linear combination of products of Kronecker deltas. For fourth order,

$$C_{ijkl} = a\, \delta_{ij}\delta_{kl} + b\, \delta_{ik}\delta_{jl} + c\, \delta_{il}\delta_{jk}. \tag{28}$$

Swapping $i \leftrightarrow j$ exchanges b and c, so minor symmetry forces $b = c$. Set $\lambda := a$ and $\mu := b$. Major symmetry then holds automatically, and contracting with $\boldsymbol{\varepsilon}$ yields the stated forms for $\boldsymbol{\sigma}$ and ψ. $\square$

Remark: Alternate moduli and forms

Isotropic materials are uniquely determined by two moduli, and any two determine the rest:

	from (λ, μ)	from (K, μ)	from (E, ν)
λ	λ	$K - \frac{2}{3}\mu$	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$
μ	μ	μ	$\dfrac{E}{2(1+\nu)}$
K	$\lambda + \frac{2}{3}\mu$	K	$\dfrac{E}{3(1-2\nu)}$
E	$\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\dfrac{9K\mu}{3K+\mu}$	E
ν	$\dfrac{\lambda}{2(\lambda+\mu)}$	$\dfrac{3K-2\mu}{2(3K+\mu)}$	ν

Hooke's law and energy decouple into volumetric and deviatoric terms using K and μ,

$$\sigma_m = K\,\varepsilon_v, \qquad \boldsymbol{s} = 2\mu\,\boldsymbol{e}, \qquad \psi = \tfrac{1}{2}\,K\,\varepsilon_v^2 + \mu\,\boldsymbol{e} : \boldsymbol{e}. \tag{29}$$

Note that $\psi > 0$ for every $\boldsymbol{\varepsilon} \neq \boldsymbol{0}$ if and only if $K > 0$ and $\mu > 0$, equivalently $E > 0$ and $-1 < \nu < \frac{1}{2}$. We will use (29) in the lecture on plasticity.

Remark: Symmetry and potentials

A recurring theme throughout this course is that a potential exists if and only if the underlying operator is symmetric. (i) Algebraic: the linear map $\boldsymbol{\varepsilon} \mapsto \mathbb{C} : \boldsymbol{\varepsilon}$ on the six-dimensional space of symmetric tensors. Major symmetry says this map is self-adjoint with respect to the inner product $\boldsymbol{A} : \boldsymbol{B}$, hence ψ exists. (ii) Integral: Betti's reciprocity theorem is a consequence of the existence of the strain-energy functional. (iii) Differential: a system of PDEs with boundary conditions descends from a functional if and only if its linearization is formally self-adjoint. (iv) Kinetic: Onsager reciprocity is the condition for a dissipation potential to exist. We will discuss (ii) and (iv) later in the course.