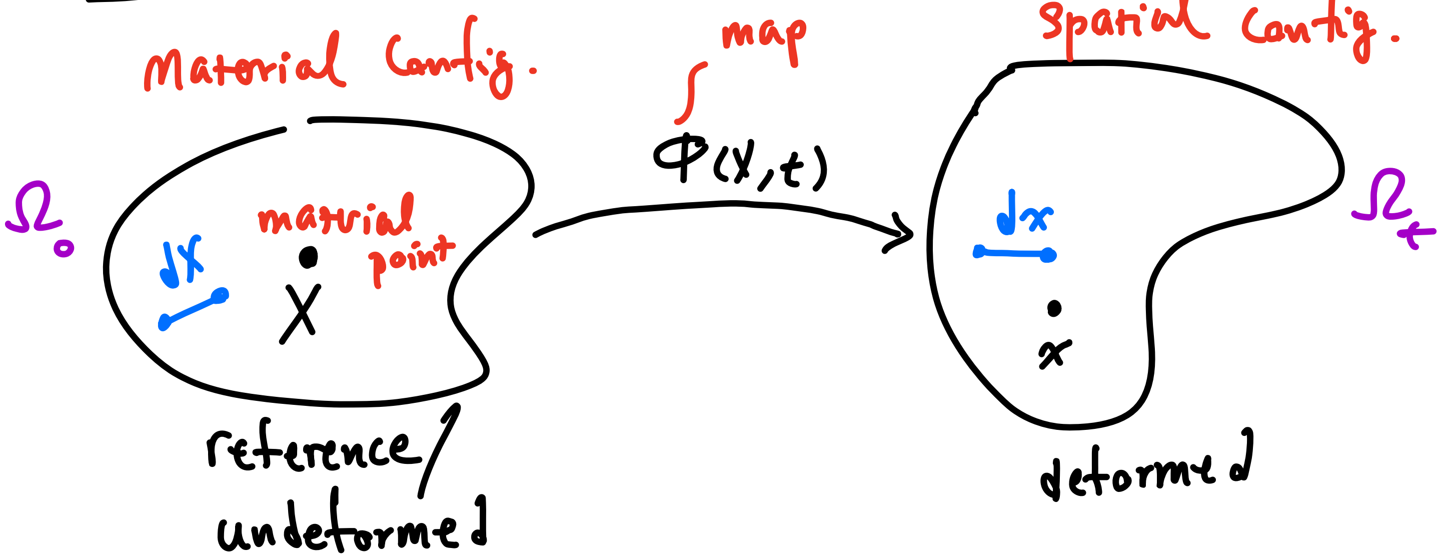
Take survey, we would appreciate your feedback.

Finite-Strain Mechanics:

So far, small strains.

1. Kinematics & deformation map



$$x = \varphi(X,t) \quad , \quad \Omega_t = \varphi(\Omega_0,t)$$

Displacement:

$$u \equiv x - X = \varphi(X,t) - X$$

Deformation gradient:

$$F \equiv \nabla_X \varphi = \frac{\partial \varphi}{\partial X} \qquad \left(F_{ij} = \frac{\partial \varphi_i}{\partial X_j} \right)$$

line segments : $\boxed{dx = F\,dX} = d\varphi = dx$

$F = \frac{\partial \varphi}{\partial X}$

$J = \det F$, $v = J V > 0$

current vol — v; ref. vol. — V

~~$J < 0 \to$ inverted~~
~~$J = 0 \to$ destroyed~~

<u>Strain</u>

$dx \cdot dx = dx^T dx$

$$|dx|^2 = dx \cdot dx = \langle dx, dx \rangle$$
$$= \langle F\,dX, F\,dX \rangle$$
$$= \langle dX, F^T F\,dX \rangle$$
$$= dX \cdot \underbrace{F^T F}_{C} dX$$

C Cauchy–Green tensor

$$\boxed{C = F^T F}$$

$$\boxed{E \equiv \tfrac{1}{2}(C - 1)}$$

Green–Lagrange Strain

<u>Relation to ϵ</u> :

$F = \frac{\partial \varphi}{\partial X}$, $u = \varphi - X$

$$\frac{\partial u}{\partial X} = F - 1$$

$$\rightarrow \boxed{F = 1 + \frac{\partial u}{\partial X}}$$

$$\rightarrow E = \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial X} + \frac{\partial u}{\partial X}^T\right.}_{=\,\epsilon} + \underbrace{\cancel{\frac{\partial u}{\partial X}\frac{\partial u}{\partial X}^T}}_{\text{Quadratic}}\left.\right)$$

Quadratic $\simeq 0$ if strain is small

if strain is small $E \simeq \epsilon$.

Polar decomposition

$$\boxed{F = \underbrace{R}_{\text{rotation}}\underbrace{U}_{\text{stretch}}}$$

$R^T R = R R^T = I$

$U = U^T$, positive def.

$U = \sqrt{C}$

Side note: $C^{\frac{1}{2}} = F^T F = W \Lambda^{\frac{1}{2}} W^T$

$$= J$$

$$\det R = \det F U^{-1} = \det F / \det U = 1$$

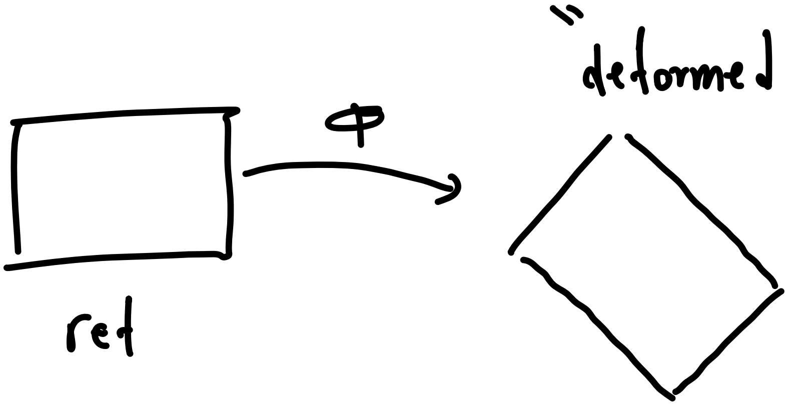
J

<u>Ex</u>: Consider solid-body rotation $\phi = QX$

rotation

$Q^T Q = Q Q^T = I$

$\det Q = 1$



$$F = \frac{\partial \phi}{\partial X} = Q, \quad C = F^T F = Q^T Q = I$$

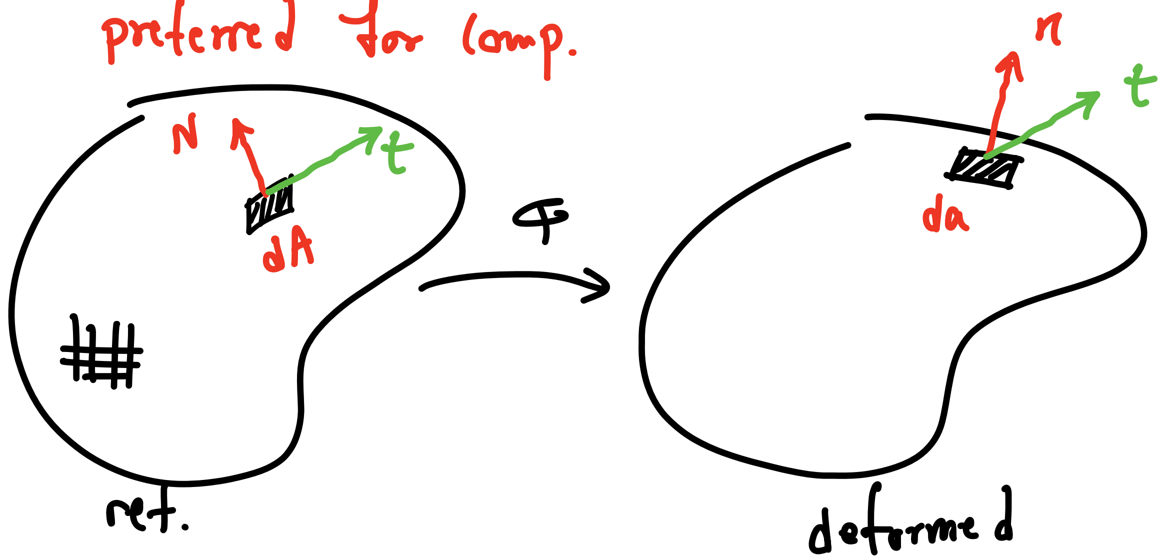
$$E = \frac{1}{2}(C - 1) = 0 \quad \text{no strain.}$$

$$u = \phi - X = QX - X = (Q - I)X$$

$$\epsilon = \left(\frac{\partial u}{\partial X} + \frac{\partial u}{\partial X}^T\right)/2 \qquad \frac{\partial u}{\partial X} = Q - I$$

$$\rightarrow \epsilon = \frac{Q + Q^T}{2} - I \neq 0$$

Stress



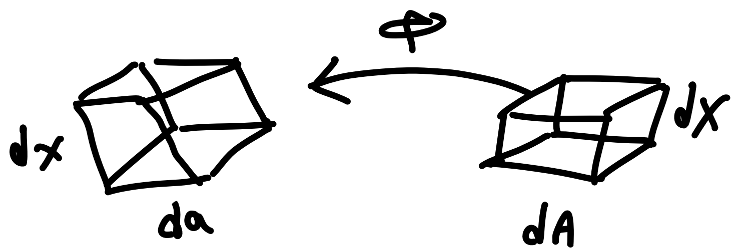
Cauchy Thm : $t = \sigma n$

Cauchy stress

Nanson's formula

$$\boxed{n\,da = \underline{\underline{J F}}^{-T} N\,dA}$$

pf.



$$dv = J\,dV$$

$$dv = dx \cdot n\,da \qquad dV = dX \cdot N\,dA$$

$$dx = F\,dX$$

$$\longrightarrow \quad dX \cdot n\,da = J F^{-1} dx \cdot N\,dA$$

$$= \langle F^{T} dx, J N\,dA \rangle$$

T

$$= dX \cdot J F^T N dA$$

$$\rightarrow \quad \underbrace{(n da - F^T N J dA)}_{\overset{!}{=} 0} \cdot dX = 0 \quad \forall dX$$

$$\square$$

Piola's Stresses

$$t da = \underbrace{\sigma n da}_{\text{def.}} = \underbrace{J \sigma F^{-T}}_{= P} \underbrace{N dA}_{\text{ref.}}$$

P first Piola stress $\left[\dfrac{\text{force current}}{\text{area ref}}\right]$

symmetric

$$S \equiv F^{-1} P \quad \text{second Piola stress} \quad \left[\dfrac{\text{force ref}}{\text{area ref}}\right]$$

$$= J F^{-1} \sigma F^{-T}$$

Power (internal)

$$D = \cancel{\sigma : \dot{\epsilon}} + \cdots \geqslant 0$$

$$\boxed{P : \dot{F} = S : \dot{E}} \quad \left[\dfrac{\text{work rate current}}{\text{volume ref}}\right]$$

pf.

$$P:\dot F = FS:\dot F = \langle FS, \dot F\rangle$$

$$= \underbrace{S}_{\text{sym}} : \underbrace{F^T\dot F}_{\text{sym}(F^T\dot F)}$$

$$\text{sym}(F^T\dot F) = \frac{F^T\dot F + \dot F^T F}{2} = \dot E$$

$$E = \frac{1}{2}(C-1)$$

$$\dot E = \frac{1}{2}\dot C, \quad C = F^T F$$

$$\dot C = \dot F^T F + F^T \dot F$$

$\square$

Def. Hyperelastic material

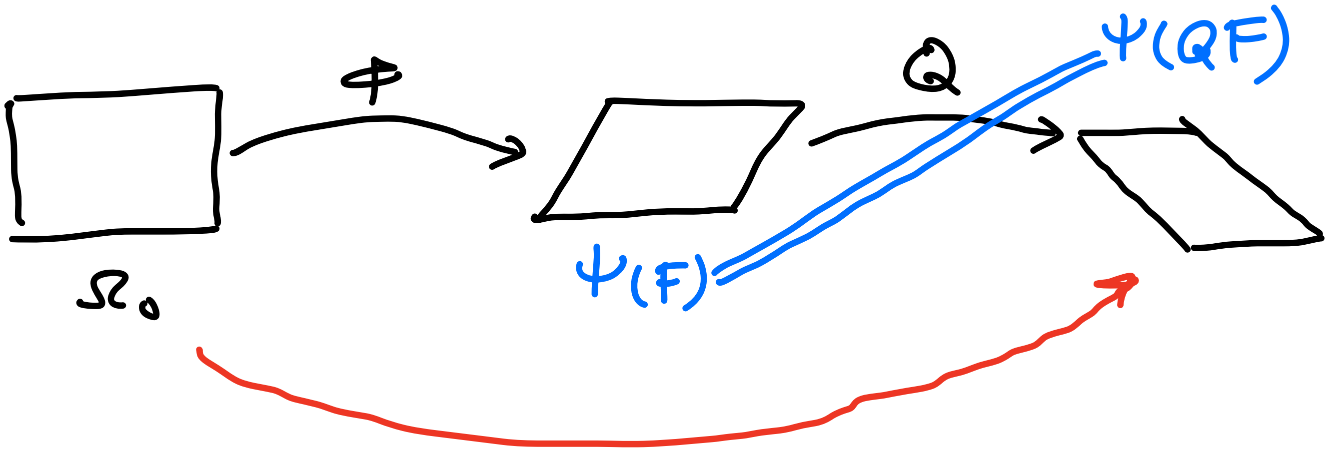
$$\Psi = \Psi(F) \longrightarrow \dot\Psi = \frac{\partial \Psi}{\partial F} : \dot F$$

$$\mathcal{D} = P:\dot F - \dot\Psi \geq 0$$

$$\longrightarrow \quad \mathcal{D} = \underbrace{\overset{0=}{\left(P - \frac{\partial\Psi}{\partial F}\right)}}_{\neq f(\dot F)} : \dot F \geq 0$$

$$\to \boxed{P = \frac{\partial \Psi}{\partial F}, \quad D = 0}$$

<u>Axiom</u>:



$$\Phi_{tot} = Q\Phi \to F_{tot} = QF$$

$$\boxed{\Psi(F) = \Psi(QF) \quad \forall Q}$$

frame indifference

Then, $\boxed{\Psi(F) = \hat{\Psi}(\underbrace{C}_{F^TF})}$ must hold.

<u>Pf.</u> $F = RU$, $Q = R^T = R^{-1}$

$$\Psi(F) = \Psi(R^T F) = \Psi(\underset{\underset{\sqrt{C}}{\parallel}}{U}) = \hat{\Psi}(C)$$

$\square$

Consequence: Given $\Psi(C)$

$$\boxed{S = 2\frac{\partial \Psi}{\partial C}}, \quad P = FS$$

pf. Outline

$$\dot{\Psi} = \frac{\partial \Psi}{\partial C} : \underset{\underset{2\dot{E}}{\parallel}}{\dot{C}} = \boxed{2\frac{\partial \Psi}{\partial C} : \dot{E}}$$

$$\mathcal{D} = 0$$

$$\parallel$$

$$S : \dot{E} - \dot{\Psi} \longrightarrow \dot{\Psi} = \boxed{S : \dot{E}}$$

$\square$

Ex: Neo-hookean solid

$$\Psi(C) = \frac{\mu}{2}(\underset{\underset{\mathrm{tr}\, C}{\parallel}}{I_1} - 3) - \mu \ln \underset{\underset{\sqrt{\det C}}{\parallel}}{J} + \frac{\lambda}{2}(\ln J)^2$$

$$J = \det F = \sqrt{\det C}$$

$$\det C = \det(F^T F) = \det(F^T)\det F = J^2$$

at small strain :

$$\underset{\sigma}{S} \simeq \lambda (\mathrm{tr}\,\epsilon) 1 + 2\mu \epsilon$$

Outlook on inelasticity

- $\epsilon = \epsilon_e + \epsilon_p$ additive decomp.
- $F = F_e F_p$ multiplicative decomp.

$$C_e = F_e^T F_e$$

$$\Psi = \Psi_e(C_e) + \Psi_h(\alpha)$$

This reduces to additive decomp at small strain :

$$F_e = 1 + H_e \ , \ F_p = 1 + H_p$$

$$\rightarrow F = F_e F_p = 1 + \underbrace{H_e + H_p}_{\text{“}\frac{\partial u}{\partial x}\text{”}} + \cancel{H_e H_p}$$

0 small strain

$$\mathcal{D} = P : \dot{F} - \dot{\psi} = S : \dot{E} - \dot{\psi} \geq 0$$

Ex: Viscoelastic solid

$$\psi = \psi(C, \dot{C}) \longrightarrow \dot{\psi} = \frac{\partial \psi}{\partial C} : \underbrace{\dot{C}}_{= 2\dot{E}} + \frac{\partial \psi}{\partial \dot{C}} : \ddot{C}$$

$f(\dot{E})$ =

$$\mathcal{D} = \underbrace{\left(S - 2\frac{\partial \psi}{\partial C} \right)}_{= S_v} : \dot{E} - \underbrace{\frac{\partial \psi}{\partial \dot{C}}}_{= 0} : \ddot{C} \geq 0$$

$$\longrightarrow \boxed{\begin{array}{l} \mathcal{D} = S_v : \dot{E} \geq 0 , \quad \dfrac{\partial \psi}{\partial \dot{C}} = 0 \\ S = 2\dfrac{\partial \psi}{\partial C} + S_v \end{array}}$$

phase-field model

$$\Lambda[u, d] = \int_\Omega g(d)\, \psi_0(\underset{C}{\cancel{\varepsilon}}) + G_c \int_\Omega \gamma - W_{ext}$$

$$\delta \Lambda = 0 \Longrightarrow \text{E-L Eqs.}$$

$$\Psi_0 = \Psi^+ + \Psi^-, \quad \Psi = g(d)\Psi^+ + \Psi^-$$

Volumetric – isochoric split

$$\underbrace{C}_{} = \underbrace{J^{2/3}}_{\substack{\det F \\ \text{volumetric}}} \underbrace{\bar{C}}_{\text{isochoric/volume-preserving}}$$

$$\bar{C} := J^{-2/3} C$$

$$\Psi_0(C) = \Psi_{vol}(J) + \Psi_{iso}(\bar{C})$$

$$\Psi(d, C) = \begin{cases} g(d)\left[\Psi_{vol} + \Psi_{iso}\right] & J \geq 1 \quad \text{exp.} \\ g(d)\Psi_{iso} + \Psi_{vol} & J < 1 \quad \text{contraction} \end{cases}$$

END OF DAY 5

END OF COURSE