Plasticity Equations Recap :

$$\psi = \psi(\epsilon, \epsilon_p, \alpha) = \psi_e(\underset{\substack{\| \\ \epsilon - \epsilon_p}}{\epsilon_e}) + \psi_h(\alpha)$$

$$\mathcal{D} = \sigma : \dot{\epsilon}_p + A\dot{\alpha} \geq 0 \quad , \quad \sigma = \frac{\partial \psi}{\partial \epsilon_e} = -\frac{\partial \psi}{\partial \epsilon_p}$$

$$A = -\frac{\partial \psi}{\partial \alpha}$$

$$E = \{(\sigma, A) \mid f(A, \sigma) \leq 0\}$$ yield function + elastic region

plastic multiplier

$$\dot{\epsilon}_p = \dot{\gamma}\frac{\partial g}{\partial \sigma} \quad , \quad \dot{\alpha} = \dot{\gamma}\frac{\partial g}{\partial A}$$

$g =$ plastic potential

$g = f$ associative flow

$$\dot{\gamma} \geq 0 , \quad f \leq 0 , \quad \dot{\gamma} f = 0$$ KKT

$\dot{\gamma} > 0 \leftrightarrow f = 0$

$\dot{\gamma} = 0 \leftrightarrow f < 0$

$$\dot{\gamma} = \frac{\frac{\partial f}{\partial \sigma} : \mathbb{C} : \dot{\epsilon}}{\frac{\partial f}{\partial \sigma} : \mathbb{C} : \frac{\partial g}{\partial \sigma} + \underset{\substack{\| \\ \frac{\partial f}{\partial A} \frac{\partial^2 \psi^h}{\partial \alpha^2} \frac{\partial g}{\partial A}}}{H}}$$

from consistency condition $\dot{f} = 0$

4. J_2 Plasticity (von Mises 1913)

Plastic deformation occurs due to the deviatoric part of stress:

$$\sigma = \underbrace{\sigma_m}_{= \frac{\mathrm{tr}\,\sigma}{3}} \mathbf{1} + \underbrace{S}_{\text{deviatoric stress}} , \qquad \mathrm{tr}\, S = 0$$

$$|S| \equiv \sqrt{S:S} \equiv \sqrt{2 J_2} \longrightarrow J_2 = \frac{1}{2} S:S$$

$$Q = \frac{S}{|S|} , \quad |Q| = 1 , \quad \mathrm{tr}\, Q = 0$$

Yield surface/function

$$A = -\frac{\partial \psi^h}{\partial \alpha}$$

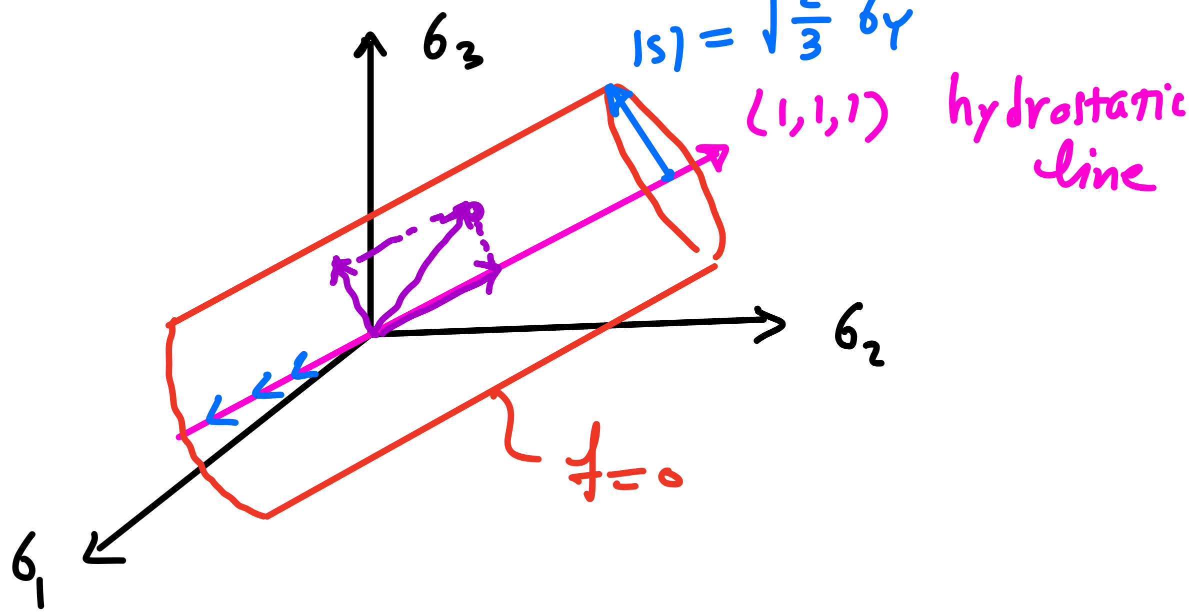
$$f(\sigma, A) = \sqrt{3 J_2} - \underbrace{(\sigma_{Y_0} - A)}_{= \sigma_Y} \leq 0$$

$g = f$ associative flow

$$\underbrace{\sqrt{3 J_2}}_{= \sigma_Y} = \sqrt{3} \, \frac{|S|}{\sqrt{2}} = \sqrt{\frac{3}{2}} |S|$$

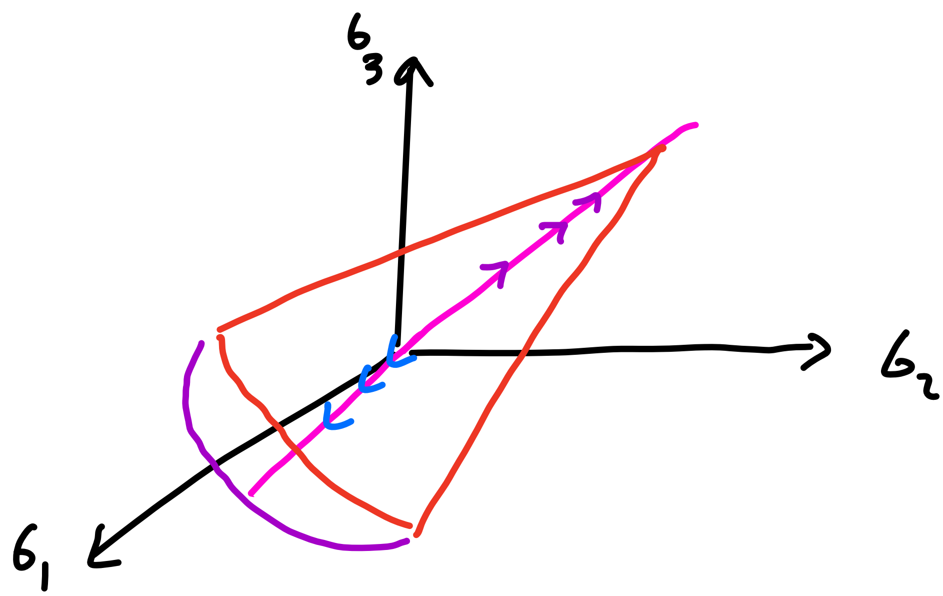
$\sqrt{2}$ /

Metals :



$$\sigma = \sigma_m (1,1,1) + \underline{\underline{(s_1, s_2, s_3)}}$$

other :



Ex: Uniaxial tension & Pure shear (1D)

tension :

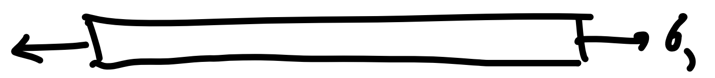
$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_m = \frac{\mathrm{tr}\,\sigma}{3} = \frac{\sigma_1}{3}$$

$$s = \sigma - \sigma_m 1 = \begin{pmatrix} \frac{2}{3}\sigma_1 & & 0 \\ & -\frac{\sigma_1}{3} & \\ 0 & & -\frac{\sigma_1}{3} \end{pmatrix}$$

$$J_2 = \frac{1}{2}\, s : s = \frac{1}{2}\left(\frac{4}{9}\sigma_1^2 + \frac{\sigma_1^2}{9} + \frac{\sigma_1^2}{9}\right)$$

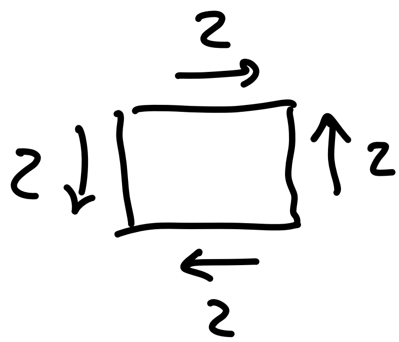
$$= \frac{1}{\cancel{2}}\, \frac{\cancel{6}^{\,2}}{\cancel{9}_{\,3}}\, \sigma_1^2 = \frac{1}{3}\sigma_1^2$$

$$\rightarrow \boxed{\sigma_1 = \underbrace{\sqrt{3J_2} = \sigma_Y}_{f=0}} \quad \sigma_1$$



Shear :

$$\sigma = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$s = \sigma , \quad J_2 = \frac{1}{2}\, s : s = \frac{1}{2}\left(\tau^2 + \tau^2\right)$$

$$\rightarrow J_2 = \tau^2 \rightarrow \tau_Y = \frac{\sqrt{J_2}\,\sqrt{3}}{\sqrt{3}} = \frac{\sigma_Y}{\sqrt{3}}$$

$$\sqrt{J_2}\,\sqrt{3} = \sigma_Y$$

$$\boxed{\tau_Y = \frac{\sigma_Y}{\sqrt{3}}}$$

Flow rule :

$$\dot{\epsilon}_p = \dot{\gamma}\,\frac{\partial f}{\partial \sigma}, \quad \dot{\alpha} = \dot{\gamma}\,\frac{\partial f}{\partial A} = 1$$

$\sqrt{\frac{3}{2}}Q$

$Q = \frac{s}{|s|}, \quad |Q| = 1$

$f(\sigma, A) = \sqrt{3J_2} - (\sigma_{Y_0} - A)$

$\rightarrow \quad \dot{\epsilon}_p = \dot{\gamma}\sqrt{\frac{3}{2}}\,Q, \quad \dot{\alpha} = \dot{\gamma}$

<u>Plastic Multiplier</u>:

$\epsilon = \epsilon_v 1 + e$

deviatoric

$$\dot{\gamma} = \frac{\sqrt{6}\,\mu\,(Q : \dot{e})}{3\mu + H}$$

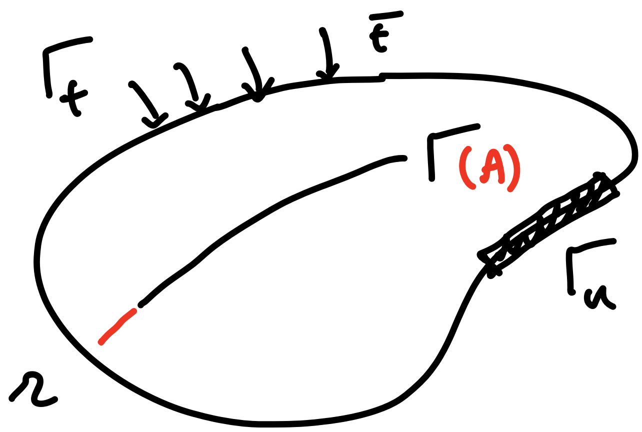
$\frac{\partial^2 \psi_h}{\partial \alpha^2}$

shear modulus (of $\mathbb{C}$)

<u>Fracture & Phase-field method</u>

1. <u>Griffith's view of fracture</u> (1921)

energy released = Cost of creating the crack



Def. Potential energy & Energy release rate

equilibrium

$$\Pi(A) \equiv \int_{\Omega\setminus\Gamma(A)} \psi(\epsilon) - \int_{\Omega} b\cdot u - \int_{\Gamma_t} \bar{t}\cdot u$$

A = crack area

u, ε = solution to equilibrium in momentum

trajectory is known ⟺ parameterize Γ(A)

$$G \equiv -\frac{d\Pi}{dA}$$

Energy release rate

trampolene

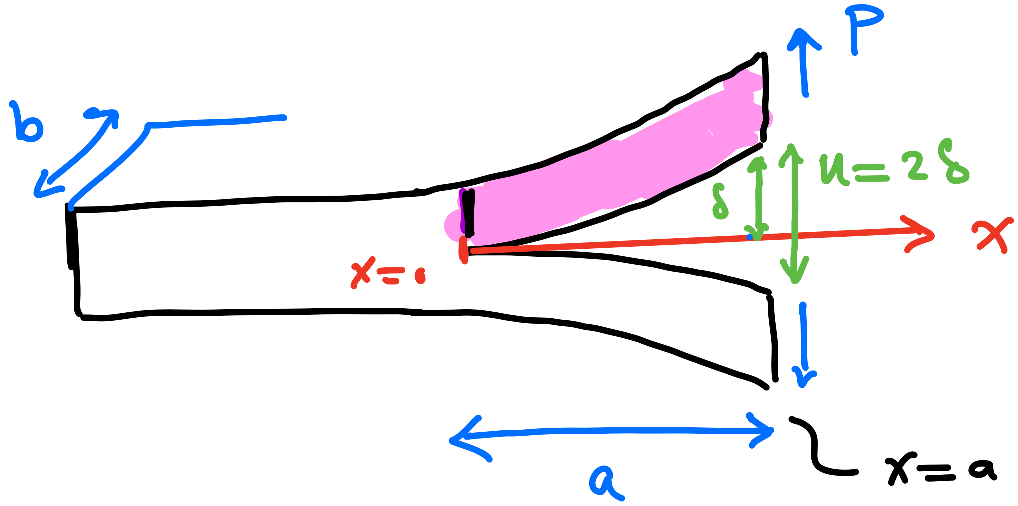
Ex:

u=0 on ∂Ω

$$\Pi(A) = \int_{\Omega\setminus\Gamma(A)} \psi \, dV = U(A)$$

$$G = -\frac{dU}{dA}$$

Ex: Double Cantilever Beam (DCB)



① Load Control:

$$EI\,w'''' = 0 \quad , \quad BCs = \begin{cases} \boxed{w(0)=0,\ w'(0)=0} \\ \text{clamped} \\ M(a)=0,\ V(a)=P \end{cases}$$

4th order.

$M(a) = EI\,w''(a)$ → $\boxed{w''(a)=0}$

$V(a) = -M'(a) = -EI\,w'''(a)$ → $\boxed{-EI\,w'''(a) = P}$

$$w(x) = \frac{P}{6EI}\left(3ax^2 - x^3\right)$$

at $x = a$

$$\rightarrow \delta = w(a) = \frac{Pa^3}{3EI}$$

$$\rightarrow \boxed{u = 2\delta = \frac{2Pa^3}{3EI}} \rightarrow \boxed{u = C(A)\,P}$$

Compliance

$$C(A) = 2a^3/3EI$$

$A = b\,a$

$$\Pi = \underbrace{U}_{=\frac{1}{2}\sum_i P_i u_i \,=\, \frac{Pu}{2}} - \underbrace{W_{ext}}_{=\,Pu} \longrightarrow \Pi = -\frac{Pu}{2}$$

$$\longrightarrow \boxed{\Pi(A) = -C(A)\,\frac{P^2}{2}}$$

$$G = -\frac{d\Pi}{dA} = \frac{dC}{dA}\,\frac{P^2}{2}$$

$$\longrightarrow \boxed{G = \frac{P^2 a^2}{bEI}} \longrightarrow G \propto a^2$$

foreshadowing

$G \leq G_c$

$G = G_c$ cracks grows

NB. $a \to 0$, $G \to 0$

② Displacement Control : $P = u / C(A)$

$$\boxed{G = \frac{9EI\,u^2}{4\,b\,a^4}} \longrightarrow G \propto \frac{1}{a^4}$$

$a \to 0$, $G \to \infty$

Griffith's Criterion for <u>stable</u> fracture

$$\dot{A} \geqslant 0 \quad , \quad G - G_c \leqslant 0 \quad , \quad (G - G_c)\dot{A} = 0$$

fracture toughness
(material property)

these are KKT conditions (w/ a threshold G_c).

<u>Meaning</u>:

① if $G < G_c \implies \dot{A} = 0$ crack does not grow

② if $G = G_c \implies \dot{A} > 0$ stable crack growth

③ if $G > G_c \implies$ Inadmissible (crack growth is unstable)

Dissipation inequality is satisfied:

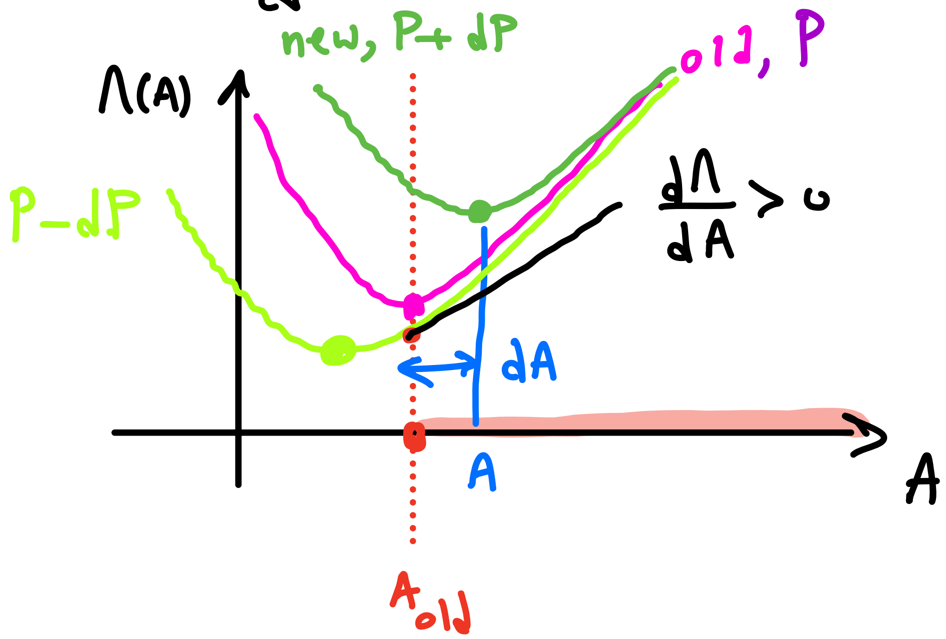
$$\mathcal{D} = \underbrace{G}_{F}\underbrace{\dot{A}}_{J} = \underbrace{G_c}_{>0}\underbrace{\dot{A}}_{\geq 0} \geqslant 0$$

Griffith as energy minimization

$$\min_{A \geq A_{old}} \Lambda(A) \quad , \quad \Lambda(A) \equiv \Pi(A) + G_c A$$

realize "A" is the minimizer

This is equivalent to KKT (Caveat!):



loading

① $\dfrac{d\Lambda}{dA} = \underbrace{\dfrac{d\Pi}{dA}}_{= -G} + G_c = -G + G_c = 0$

$\longrightarrow \boxed{G = G_c}$ crack grows stably

unloading

② $\dfrac{d\Lambda}{dA} = -G + G_c > 0 \longrightarrow \boxed{G < G_c}$

crack stays frozen

Key Problem : Above picture requires
$\Gamma(A)$ to be parameterized
$\equiv$ crack trajectory is known

Generalizing to arbitrary crack trajectory :

$$\Lambda[u,\Gamma] = \int_{\Omega\setminus\Gamma} \psi \, dv - \int_{\Omega} b.u - \int_{\Gamma_t} \bar{t}.u + G_c \underbrace{|\Gamma|}_{= A}$$

and

$$\min_{\substack{u \\ \Gamma \supseteq \Gamma_{old}}} \Lambda[u,\Gamma] = \min_{\Gamma \supseteq \Gamma_{old}} \min_{u} \Lambda[u,\Gamma]$$

unconstrained

$$= \min_{\substack{\cancel{\Gamma \supseteq \Gamma_{old}} \\ A > A_{old}}} \Lambda[u_{eq}(A), \Gamma(A)]$$

if crack path were known

The minimization seeks a global min.
(KKT $\Longleftrightarrow$ local search around A_{old})

problem: nucleation Not captured

Ex: Crack Nucleation in DCB

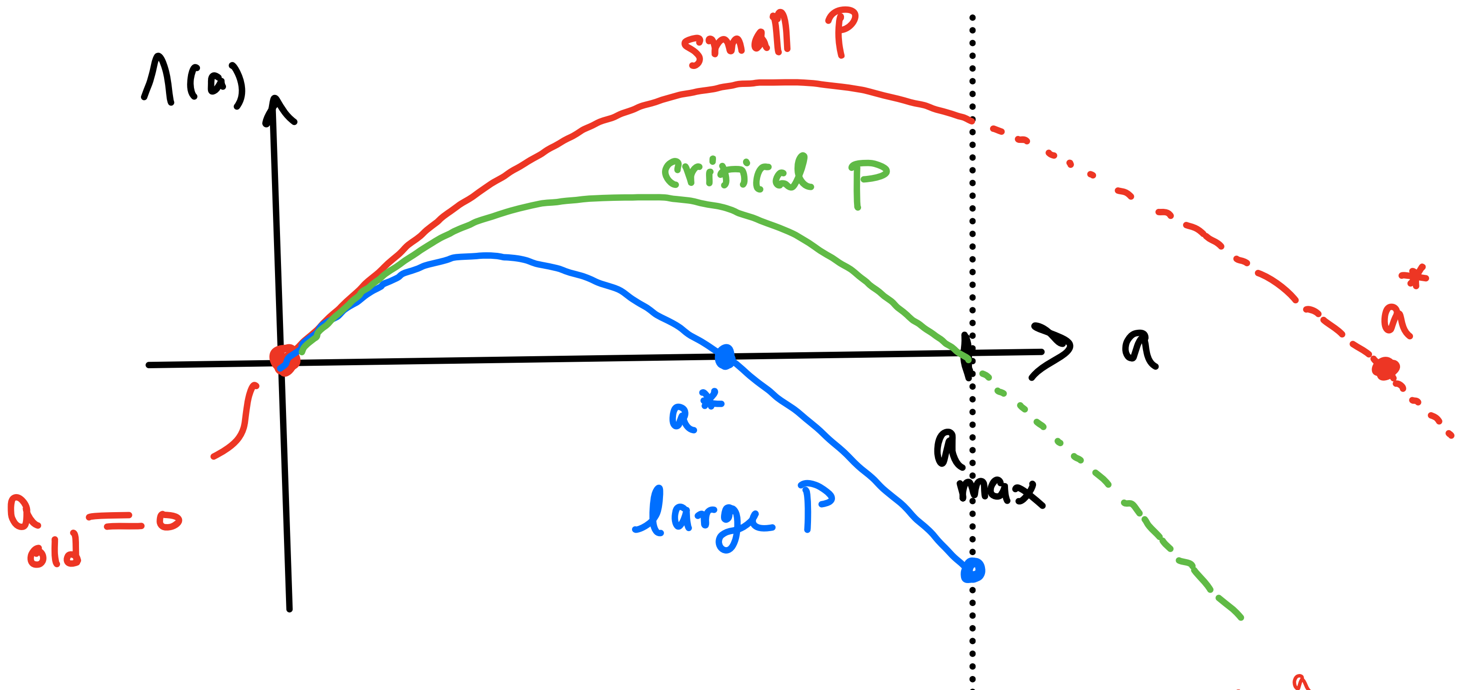
① Load Control

$$\Lambda = \Pi + G_c A$$

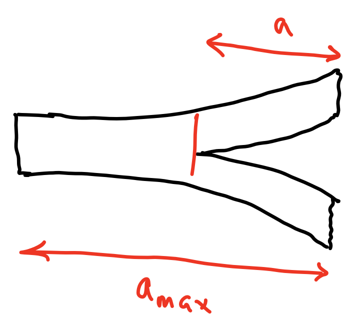
$$\rightarrow \quad \Lambda(a) = -\frac{P^2 a^3}{3EI} + G_c b a$$

Suppose an intact initial state $(a=0)$

notice $\Lambda(0) = 0$.



- small P $\longrightarrow$ $a=0$ minimizer
- large P $\longrightarrow$ $a = a_{max}$ minimizer

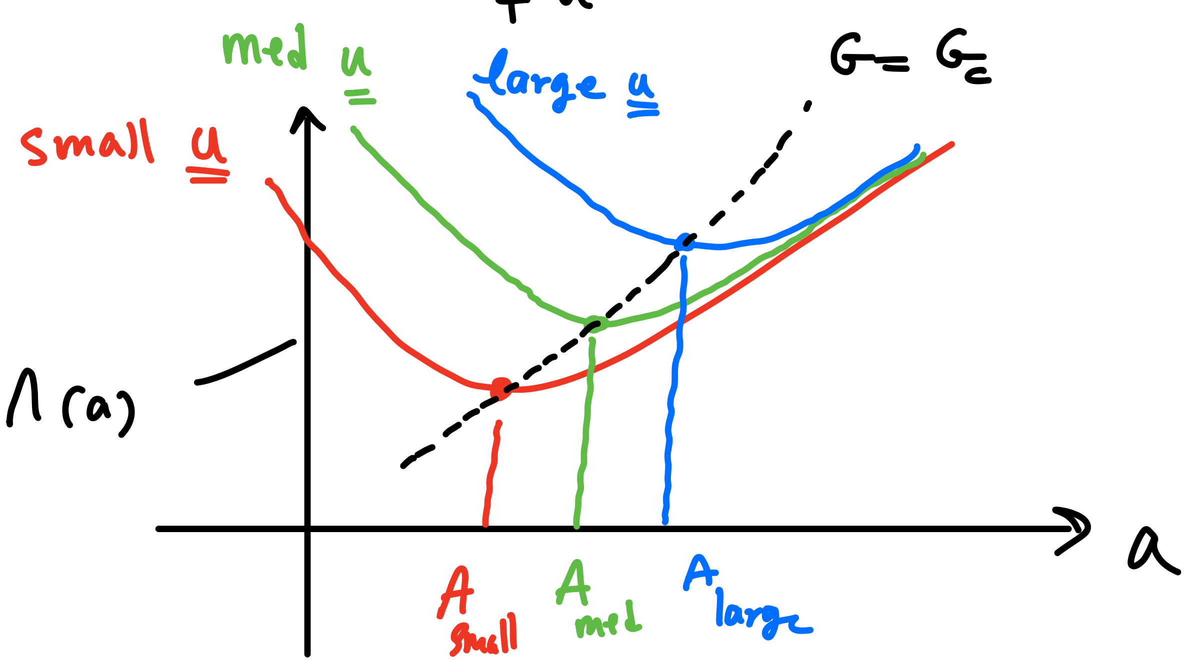


For small perturbations around P_{crit}, $\underline{\underline{a}}$ goes from $\underline{0}$ to $\underline{\underline{a_{max}}}$. $\Longrightarrow$ unstable crack growth.

$\underline{NB}$: $a \to 0$, $G \to 0$ $\longrightarrow$ local search predicts no crack b/c $G < G_c$

② <u>Displacement Control</u>

$$\Lambda(a) = \frac{3EIu^2}{4a^3} + G_c b a$$



minima $\longrightarrow$ Continuous change

$$a_{eq} = \left(\frac{9EIu^2}{4bG_c}\right)^{1/4} \propto \sqrt{u}$$

2. Phase-field model of fracture

minimizing over Γ (crack set) is hard, but mimizing over a field (damage var) is easy.

$$\Lambda[u,\Gamma] = \dots + G_c|\Gamma|$$

$= A$

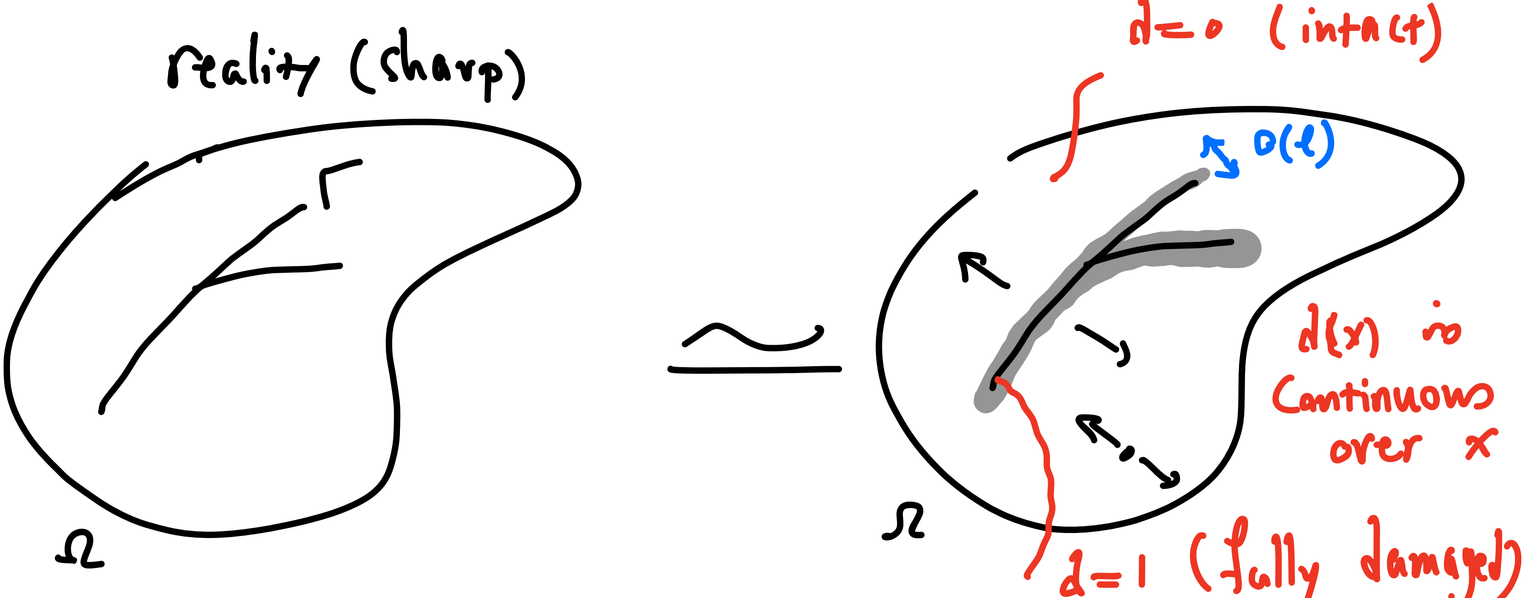
Regularizing (replacing) the $G_c|\Gamma|$ term:

AT2 = Ambriosio – Tortorelli

$$|\Gamma| \simeq \int_\Omega \gamma \, dV , \qquad \gamma(d,\nabla d) = \frac{d^2}{2\ell} + \frac{\ell}{2}|\nabla d|^2$$

crack density function

length parameter



As $\ell \to 0$, the sharp crack is recovered.

Degradation function

stored elastic energy density of intact

$$\psi(\epsilon, d) = g(d)\, \psi_0(\epsilon) \quad , \quad \psi_0(\epsilon) = \frac{1}{2}\, \epsilon : \mathbb{C} : \epsilon$$

degradation function

$$g(d) = (1-d)^2$$

$$\sigma = \frac{\partial \psi}{\partial \epsilon} = g(d) \frac{\partial \psi_0}{\partial \epsilon} = \underline{g(d)}\, \mathbb{C} : \epsilon$$

$$g(0) = 1 \quad , \quad g(1) = 0 \quad , \quad g'(d) \leq 0$$

undamaged

Def. Phase-field functional

$$g(d)\, \psi_0(\epsilon)$$

$$\Lambda[u, d] \equiv \int_\Omega \psi(\epsilon, d) + G_c \int_\Omega \gamma(d, \nabla d)$$

$\neq \Gamma$

$$- \int_\Omega b \cdot u - \int_{\Gamma_t} \bar{F} \cdot u$$

Thm. <u>Euler-Lagrange Eqs</u>

$$\delta \Lambda = 0 \iff \begin{cases} ① \ \nabla \cdot \sigma + b = 0, \quad \sigma n = \bar{t} \text{ at } \Gamma_t \\ ② \ \dfrac{G_c}{\ell}\left(d - \ell^2 \underset{\underset{\nabla^2 d}{\|}}{\Delta d}\right) = -g' \psi_0 \end{cases}$$

$(\nabla d \cdot n = 0 \text{ on } \partial\Omega)$

pf.

$$\Lambda[u, d] \equiv \int_\Omega g_{(d)} \psi_0(\epsilon) + G_c \int_\Omega \gamma(d, \nabla d)$$

$$\cancel{- \int_\Omega b \cdot u - \int_{\Gamma_t} \bar{F} \cdot u}$$

<u>$\delta_u \Lambda = 0$</u> :

$$\delta_u \Lambda = \int_\Omega \overbrace{g \frac{\partial \psi_0}{\partial \epsilon}}^{=\sigma} : \delta\epsilon - \int_\Omega b \cdot \delta u - \int_{\Gamma_t} \bar{F} \cdot \delta u = 0$$

$$\longrightarrow \quad \nabla \cdot \sigma + b = 0, \quad \sigma n = \bar{t} \text{ on } \Gamma_t$$

$\underline{\delta_d \Lambda = 0}$:

$$\delta_d \Lambda = \int_\Omega g' \psi_0 \, \delta d + G_c \int_\Omega \underline{\delta \gamma}$$

$$\ell \nabla d$$

$$\delta \gamma = \underbrace{\gamma_{,d}}_{\frac{d}{\ell}} \delta d + \overset{\ell \nabla d}{\gamma_{,\nabla d}} \, \underbrace{\delta \nabla d}_{\nabla \delta d}$$

$$\gamma = \frac{d^2}{2\ell} + \frac{\ell}{2} \underbrace{\cancel{|\nabla d|^2}}_{\nabla d \cdot \nabla d}$$

$$\longrightarrow \boxed{\int_\Omega \delta \gamma} = \underline{\underline{\int_\Omega \frac{d}{\ell} \delta d}} + \int_\Omega \ell \, \underbrace{\nabla d \cdot \nabla \delta d}_{\cancel{\int_\Omega \nabla \cdot (\nabla d \, \delta d)} - \boxed{\int_\Omega \Delta d \, \delta d}}$$

$$\int_\Omega \nabla \cdot (\nabla d \, \delta d) = \int_{\partial \Omega} \cancelto{0}{\nabla d \cdot n \, \delta d}$$

$$\delta_d \Lambda = \int_\Omega g' \psi_0 \, \underline{\delta d} + \frac{G_c}{\ell} \int_\Omega d \, \underline{\delta d} - G_c \ell \int_\Omega \Delta d \, \underline{\delta d}$$

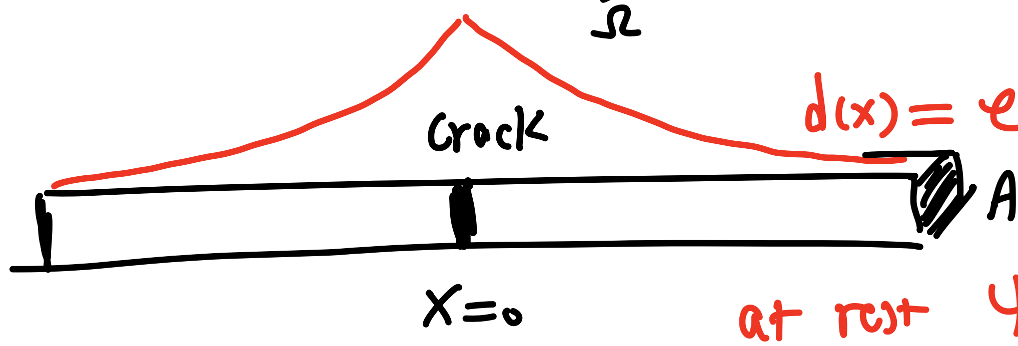
$$= \int_\Omega \underbrace{\left(g' \psi_0 + \frac{G_c}{\ell} d - G_c \ell \, \Delta d \right)}_{= 0} \delta d = 0 \quad \forall \delta d$$

$\square$

prop. 1D bar

② $A = |\Gamma| = \int_\Omega \gamma \, dV$

$d(x) = e^{-\frac{|x|}{\ell}}$ ①



at rest $\psi_0 = 0$

pf. $\cancel{\frac{G_c}{\ell}} \left(d - \ell^2 \underbrace{\Delta d}_{d_{xx}} \right) = - g' \cancel{\psi_0} \to 0$

$\to \; d - \ell^2 d_{xx} = 0 \xrightarrow{x > 0} d(x) = \underset{\to 0}{\cancel{A}} e^{\frac{x}{\ell}} + \underset{\to 1}{\cancel{B}} e^{-\frac{x}{\ell}}$

$$\rightarrow d(x) = e^{-\frac{|x|}{\ell}}$$

at $x \rightarrow \infty$
$d \rightarrow 0$

$d(x=0) = 1$

$$\int_\Omega \gamma = \int_A \int_{-\infty}^{+\infty} \gamma = \dots = \int_A dA = A$$

① <u>Crack irreversibility</u> $(\dot{A} \geq 0)$

$$\underbrace{\frac{G_c}{\ell}(d - \ell^2 \Delta d)}_{R} = \underbrace{-g' \psi_0}_{Y} = \underbrace{2(1-d)}_{\geq 0} \underbrace{\cancel{\psi_0}}_{\geq 0}\ H$$

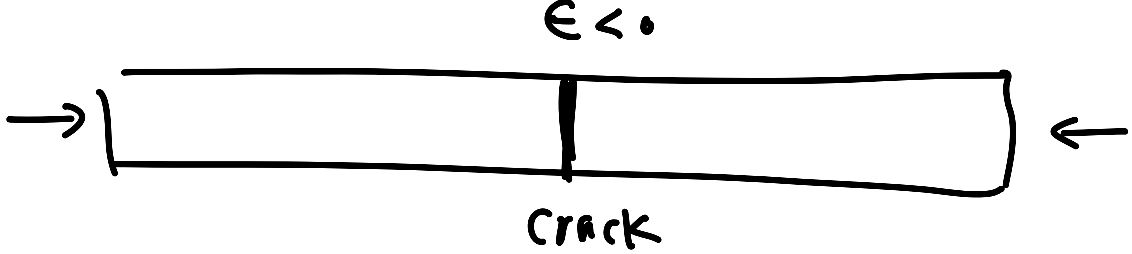
$g = (1-d)^2$

R = resistence to fracture $\qquad Y \equiv$ driving force for fracture $(\simeq \sigma)$

Replace ψ_0 with $\boxed{H(x,t) = \max_{s \leq t} \psi_0(x,s)}$

② Under Compression the sample should not "feel" broken / degraded:

$$\psi = g(d)\, \psi_0 \longrightarrow \sigma = \underline{g(d)\, \mathbb{C}} : \underline{\underline{\epsilon}}$$



$$g(1) = 0$$

$$\psi_0 = \psi^+ + \psi^-$$

$$\psi = g(d)\,\psi^+ + \psi^- \longrightarrow \sigma = g(d)\frac{\partial \psi^+}{\partial \epsilon} + \frac{\partial \psi^-}{\partial \epsilon}$$

(A) Volumetric – Deviatoric Split (isotropic $\mathbb{C}$)

$$\epsilon = \epsilon_v \mathbf{1} + e$$

$$\psi_0 = \frac{1}{2} K \epsilon_v^2 + \mu\, e : e$$

$$\langle x \rangle_+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}, \quad \langle x \rangle_- = \begin{cases} 0 & x \geq 0 \\ x & x < 0 \end{cases}$$

$$\psi_0^+ = \frac{K}{2}\langle \epsilon_v \rangle_+^2 + \mu\, e : e \quad \longleftarrow \times g(d)$$

$$\psi_0^- = \frac{K}{2}\langle \epsilon_v \rangle_-^2$$

Amor, Marigo, and Maurini (2009)

(B) Spectral split

$$\underset{\epsilon^{\mp}}{\epsilon} = \sum_i \varepsilon_i\, n_i \otimes n_i \longrightarrow \epsilon^{\pm} = \sum_i \langle \varepsilon_i \rangle^{\pm} n_i \otimes n_i$$

$$\psi_0^{\pm} = \frac{1}{2}\lambda \langle \mathrm{tr}\,\epsilon \rangle_{\pm}^2 + \mu\, \mathrm{tr}(\epsilon_{\pm}^2)$$

NB: $\psi_0 = \psi^+ + \psi^-$ $\qquad \chi_- \chi_+ = 0$

Summarize (PFM):

$$\nabla \cdot \sigma + b = 0 \quad \text{on } \Omega, \qquad \sigma n = \bar{t} \quad \text{on } \Gamma_t$$

$$\underbrace{\frac{G_c}{\ell}(d - \ell^2 \Delta d)}_{R} = \underbrace{-g' \cancel{H}^{\psi_0}}_{Y}, \qquad H = \max_{s \le t} \psi_0(x,s)$$

$$\nabla d \cdot n = 0 \quad \text{on } \partial\Omega$$

Thermodynamic perspective:

$$\dot{d} \geq 0, \quad Y - R \leq 0, \quad \dot{d}(Y - R) = 0$$

- if $Y < R \longrightarrow \dot{d} = 0$
- if $Y = R \longrightarrow \dot{d} > 0$

$g = (1-d)^2$

$Y = -g' \psi_0 = 2(1-d)\psi_0 \geqslant 0$

$$\mathcal{D} = G\dot{A} \geqslant 0 \quad , \quad \mathcal{D} = \underbrace{Y}_{\geqslant 0}\underbrace{\dot{d}}_{\geqslant 0} \geqslant 0$$

$$(G, \dot{A}) \sim (Y, \dot{d})$$

$$\frac{d\Lambda}{dA} = -G + G_c \geqslant 0$$

$$\frac{\delta\Lambda}{\delta d} = -Y + R \geqslant 0$$

Def.

$$\delta I = \int_\Omega \square \, \delta d \longrightarrow \square \equiv \frac{\delta I}{\delta d}$$

END of Day 4