Recap:

$$\begin{cases} \psi = \psi(\epsilon, \dot{\epsilon}, \xi) \\ \sigma = \sigma(\epsilon, \dot{\epsilon}, \xi) \end{cases} \longrightarrow \begin{cases} \dfrac{\partial \psi}{\partial \dot{\epsilon}} = 0 \quad \checkmark \\ \sigma = \dfrac{\partial \psi}{\partial \epsilon} + \underline{\underline{\sigma_v}} \quad \checkmark \\ \mathcal{D} = \sigma_v : \dot{\epsilon} + X \cdot \dot{\xi} \geqslant 0 \quad \checkmark \end{cases}$$

internal vars (ξ)

viscous stress (σ_v)

$X \equiv -\dfrac{\partial \psi}{\partial \xi}$ Conjugate force

$\dot{\xi}$ "flux"

$$\mathcal{D} = \sigma : \dot{\epsilon} - \dot{\psi} - s\dot{T} - \frac{q \cdot \nabla T}{T} \geqslant 0$$

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon} : \dot{\epsilon} + \frac{\partial \psi}{\partial \dot{\epsilon}} : \ddot{\epsilon} + \underline{\frac{\partial \psi}{\partial \xi}} \dot{\xi}$$

$$0 = \underbrace{(\qquad\qquad)}_{\neq f(\text{flux})} : \text{flux}$$

flux: arbitrary, controllable

Def. Dissipation Potential for the F.J pair, F = force, J = flux

$$\Phi(J) \geqslant 0, \quad \Phi(0) = 0, \quad \text{convex}, \quad F = \frac{\partial \Phi}{\partial J}$$

prop. if F.J contributes to $\mathcal{D}$, and

(F, J) come from $\Phi(J)$ (dissip. potential)

$$\Longrightarrow \quad F \cdot J \geq 0$$

Ex:

$$\mathcal{D} = \underbrace{\sigma_v}_{F} : \underbrace{\dot{\epsilon}}_{J} + \overbrace{X}^{F} \cdot \overbrace{\dot{\xi}}^{J} \geq 0$$

$$X \equiv -\frac{\partial \psi}{\partial \xi}$$

$$X = \frac{\partial \Phi}{\partial \dot{\xi}} \longrightarrow \boxed{\frac{\partial \Phi}{\partial \dot{\xi}} = -\frac{\partial \psi}{\partial \xi}}$$

Kelvin-Voigt material

$$\psi(\epsilon) = \frac{1}{2}\, \epsilon : \mathbb{C} : \epsilon$$

no internal var.

$$\Phi(\dot{\epsilon}) = \frac{1}{2}\, \eta\, \dot{\epsilon} : \dot{\epsilon}$$

$$\mathcal{D} = \sigma_v : \dot{\epsilon} \geq 0$$

$$\sigma = \frac{\partial \psi}{\partial \epsilon} + \sigma_v \longrightarrow \boxed{\sigma = \mathbb{C} : \epsilon + \eta\, \dot{\epsilon}}$$

$$\frac{\partial \psi}{\partial \epsilon} = \mathbb{C} : \epsilon \qquad \sigma_v = \frac{\partial \Phi}{\partial \dot{\epsilon}} = \eta\, \dot{\epsilon}$$

Heat Conduction

$$-\frac{q \cdot \nabla T}{T} = \underbrace{q}_{J} \cdot \underbrace{\left(-\frac{\nabla T}{T}\right)}_{F} \geq 0$$

Def. Quadratic dissipation potential

$$\Phi(J) = \frac{1}{2} J \cdot M^{-1} J \quad , \quad M = \text{positive definite}$$

tensor/scalar/...

$$F = \frac{\partial \Phi}{\partial J} = M^{-1} J$$

$$x \cdot Mx > 0 \quad \forall x$$

$$\Downarrow$$

$$y \cdot M^{-1} y > 0$$

$\rightarrow$ $\boxed{J = MF}$ linear constitutive law

$\rightarrow$ $F \cdot J = F \cdot MF \geq 0$ entropy producing

Ex: Fourier heat Conduction

$$J = q$$

$$F = -\frac{\nabla T}{T}$$

$$\Phi(q) = \frac{1}{2T} q \cdot K^{-1} q$$

$$-\frac{\nabla T}{\cancel{T}} = \frac{\partial \Phi}{\partial q} = \frac{1}{\cancel{T}} K^{-1} q$$

$\rightarrow$ $q = -K \nabla T$ familiar

$$q \cdot \left(-\frac{\nabla T}{T}\right) = (-K\nabla T)\cdot\left(-\frac{\nabla T}{T}\right)$$

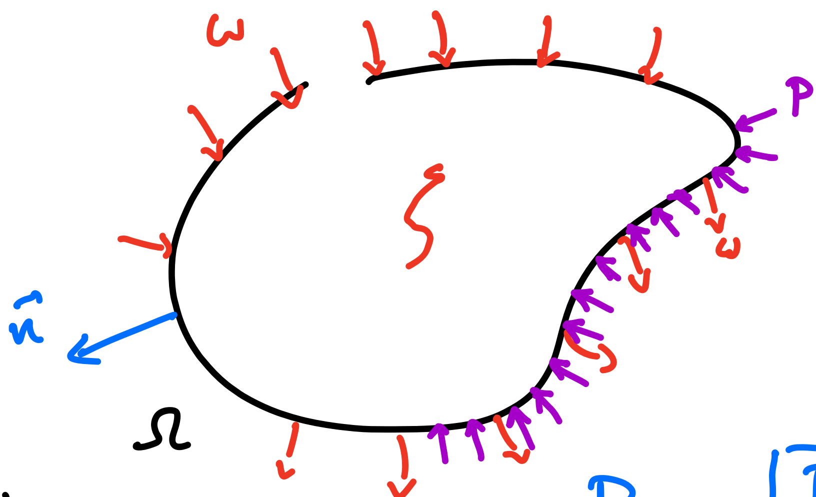
$$= \frac{\nabla T \cdot K \nabla T}{T} \geq 0 \quad \text{b/c } K \text{ positive definite}$$

5. Poroelasticity

$$\zeta \equiv \frac{\text{fluid volume change}}{\text{bulk volume}} \quad \begin{cases} + & \text{entering} \\ - & \text{exiting} \end{cases}$$

ζ: fluid content

Mass balance : $\dot{\zeta} + \nabla \cdot w = 0$



1st law

$$\frac{d}{dt}\int_\Omega e\, dv = \cancel{P_{act}} + \dot{Q}$$

$P_{act} = P_{int} + \boxed{P_{fluid}}$

$\cancel{P_{ext}}$ $\quad P_{int} = \int_\Omega \sigma : \dot{\epsilon}$

$$P_{fluid} = -\int_{\partial\Omega} P n\cdot w \, dA = -\int_{\Omega} \underbrace{\nabla\cdot(Pw)}_{\nabla P\cdot w + P \underbrace{\nabla\cdot w}_{-\dot\zeta}}\, dV$$

$$\boxed{P_{fluid} = -\int_{\Omega} (\nabla P\cdot w - P\dot\zeta)\, dV}$$

Local form of 1st law

$$\dot e = \sigma:\dot\epsilon + \underline{P\dot\zeta - \nabla P\cdot w} - \nabla\cdot q + r$$

Prop. Dissipation inequality

$$\mathcal{D} = \sigma:\dot\epsilon + \underset{F}{P}\,\underset{J}{\dot\zeta} - \dot\psi - \cancel{s\dot T} - \underset{F}{\nabla p}\cdot\underset{J}{w} - \cancel{\frac{q\cdot\nabla T}{T}} \geq 0$$

Prop. Poroelastic Constitutive law (T = cte)

$$\psi = \psi(\epsilon, \zeta)$$

$$\sigma = \sigma(\epsilon, \xi)$$

$$\rightarrow \quad \dot{\psi} = \frac{\partial \psi}{\partial \epsilon} : \dot{\epsilon} + \frac{\partial \psi}{\partial \xi} \dot{\xi}$$

$$\rightarrow \quad D = \underbrace{\overbrace{\left(\sigma - \frac{\partial \psi}{\partial \epsilon}\right)}^{0 =}}_{\neq f(\dot{\epsilon})} : \dot{\epsilon} + \overbrace{\underbrace{\left(p - \frac{\partial \psi}{\partial \xi}\right)}_{0 =}}^{\neq f(\dot{\xi})} \dot{\xi} - \nabla p \cdot w \geq 0$$

$$\rightarrow \quad \boxed{\sigma = \frac{\partial \psi}{\partial \epsilon}, \quad p = \frac{\partial \psi}{\partial \xi}, \quad D = -\nabla p \cdot w \geq 0}$$

Constitutive law assumed, must satisfy $D \geq 0$

$$\frac{\partial \sigma}{\partial \xi} = \frac{\partial p}{\partial \epsilon} = \frac{\partial^2 \psi}{\partial \epsilon \partial \xi}$$

Ex: Biot's free energy

$$\psi(\epsilon, \xi) = \frac{1}{2} \epsilon : \mathbb{C} : \epsilon + \frac{1}{2} M (\xi - b \epsilon_v)^2$$

Biot modulus

Biot coefficient

$$\boxed{p = \frac{\partial \psi}{\partial \zeta}} = \boxed{M(\zeta - b\,\epsilon_v)}$$

NB. $\frac{\partial \epsilon_v}{\partial \epsilon_{ij}} = 1_{ij}$

$$\sigma = \frac{\partial \psi}{\partial \epsilon} = \mathbb{C}:\epsilon - b\,\underbrace{M(\zeta - b\,\epsilon_v)}_{= p}\,1$$

$$\rightarrow \boxed{\sigma = \mathbb{C}:\epsilon - b\,p\,1}$$

total stress

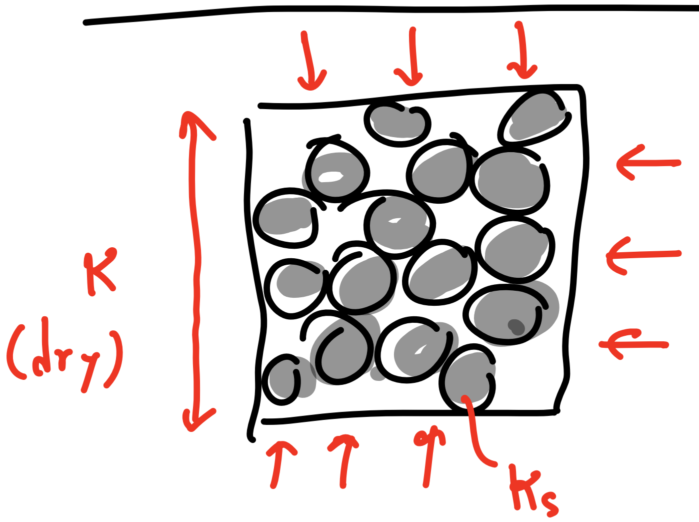
effective stress

fluid pressure contribution

Relation between (b, M) and (K, K_s)

K: dry skeleton Bulk modulus

K_s: grain bulk mod.



$K < K_s$

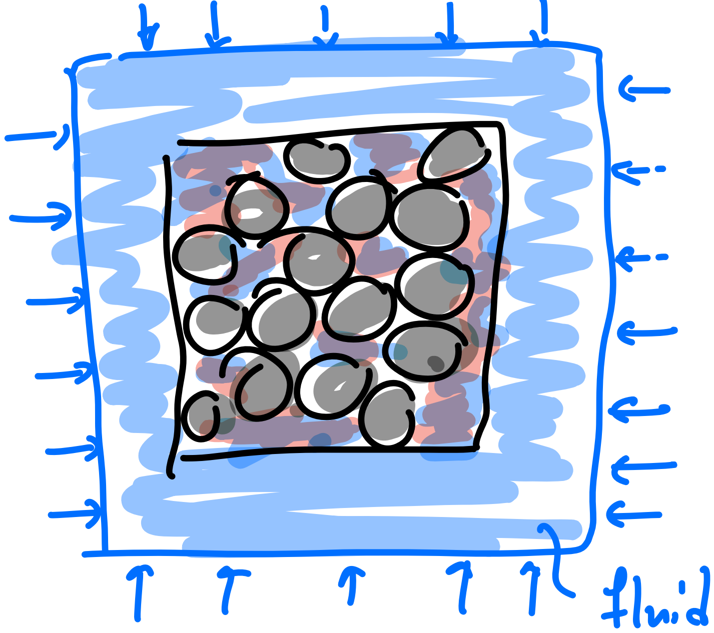
① Biot coefficient

prop. $b = 1 - \frac{K}{K_s} \le 1$

soil/loose media $K \ll K_s$

$\Longrightarrow b \simeq 1$

pf.



Consolidated rocks $\Rightarrow b<1$

unjacketed Exp.

$$\epsilon_v = -\frac{p}{K_s}$$

$$\frac{tr}{3}\left(\sigma = \underline{\mathbb{C} : \epsilon} - b\,p\,\mathbf{1}\right)$$

$$\rightarrow \underbrace{\frac{tr\,\sigma}{3}}_{-p\,=\,\sigma_m} = K \underbrace{\epsilon_v}_{=\,-\frac{p}{K_s}} - b p$$

$$\rightarrow +\cancel{p} = +\frac{K}{K_s}\cancel{p} + b\,\cancel{p}$$

$$\rightarrow \boxed{b = 1 - \frac{K}{K_s}} \qquad \square$$

② Biot's Modulus

let ϕ_f be porosity :

$$\frac{1}{M} = \frac{\phi_f}{K_f} + \frac{b - \phi_f}{K_s}$$

pf.

$$\ominus \left\{ \begin{aligned} \frac{\Delta V_{pore}}{\cancel{V_{pore}} \, V_{blk}} &= \phi_f \epsilon_v = -\frac{P\phi_f}{K_s} \quad \text{assumption} \\ \frac{\Delta V_{fluid}}{\cancel{V_{pore}} \, V_{blk}} &= \frac{\Delta V_{fluid}}{\cancel{V_{fluid}} \, V_{blk}} = -\frac{P}{K_f}\phi_f \end{aligned} \right.$$

$$\zeta = \frac{P\phi_f}{K_f} - \frac{P\phi_f}{K_s} = \boxed{P\phi_f\left(\frac{1}{K_f} - \frac{1}{K_s}\right)}$$

$$\zeta = \boxed{\frac{P}{M} + b\underbrace{\epsilon_v}_{= -\frac{P}{K_s}}}$$

$$P = M(\zeta - b\,\epsilon_v)$$

Solving for $\frac{1}{M}$, claim is proven. $\square$

Ex: $\underbrace{-\nabla p}_{F}\cdot\underbrace{\omega}_{J}$

$$\Phi(\omega) = \frac{\eta_f}{2}\,\omega\cdot K^{-1}\omega \longrightarrow -\nabla p = \frac{\partial \Phi}{\partial \omega} = \eta_f K^{-1}\omega$$

$$\longrightarrow \boxed{\omega = -\frac{K}{\eta_f}\nabla p}$$

$$\longrightarrow -\nabla p\cdot\omega = \frac{1}{\eta_f}\nabla p\cdot \underbrace{K}_{\text{positive def.}}\nabla p > 0$$

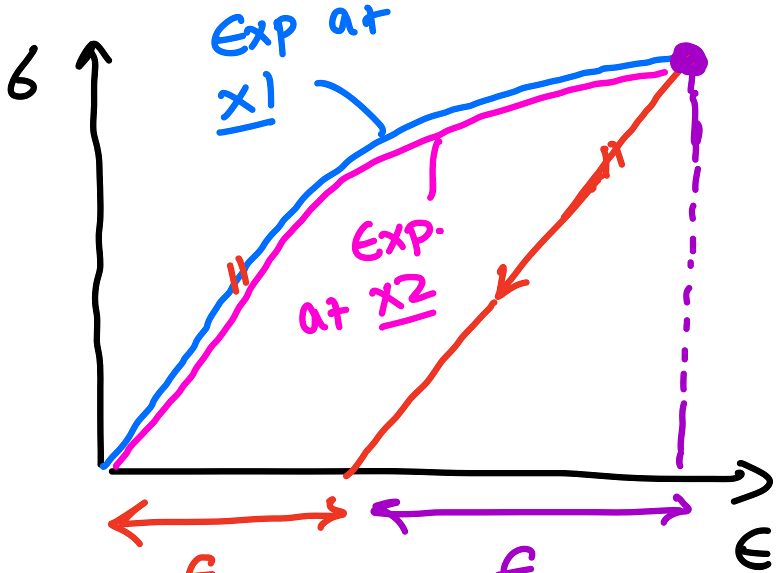
Rate-Independent Plasticity (T = ct)

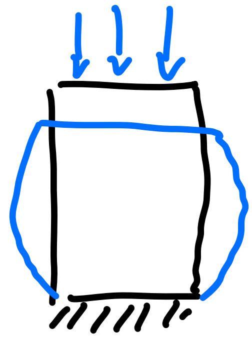
$$\mathcal{D} = \sigma:\dot{\epsilon} - \dot{\psi} \geq 0$$

$$\psi = \psi(\epsilon, \cancel{\dot{\epsilon}}, \xi)$$
$$\sigma = \sigma(\epsilon, \cancel{\dot{\epsilon}}, \xi)$$

$$\longrightarrow \sigma = \frac{\partial \psi}{\partial \epsilon} + \cancelto{0}{\sigma_v}, \quad \chi = -\frac{\partial \psi}{\partial \xi}$$

$$\mathcal{D} = \cancel{\sigma_v : \dot{\epsilon}}^{\,0} + X \cdot \dot{\xi} \geq 0$$





$$\epsilon = \epsilon_p + \epsilon_e$$

For <u>rate-independence</u>:

$$\mathcal{D} = X \cdot \dot{\xi} \geq 0 \, , \qquad \sigma = \frac{\partial \psi}{\partial \epsilon}$$

$$\psi = \psi(\epsilon, \xi) \qquad\qquad X = -\frac{\partial \psi}{\partial \xi}$$

$$\sigma = \sigma(\epsilon, \xi)$$

<u>Def</u>. <u>Internal Variables of Plasticity</u>

$$\xi = (\epsilon_p, \alpha)$$

plastic strain tensor
macroscopic + measurable

"hardening" variables = microscopic changes
(not measurable)

Def. Postulated free energy

$$\Psi(\epsilon, \epsilon_p, \alpha) = \underbrace{\Psi_e}_{\text{recoverable part of free energy}}(\underbrace{\epsilon - \epsilon_p}_{= \epsilon_e \text{ elastic strain}}) + \Psi_h(\alpha)$$

Prop. Dissipation inequality

$$\mathcal{D} = X \cdot \dot{\xi} \geq 0$$

$$\xi = \begin{pmatrix} \epsilon_p \\ \alpha \end{pmatrix} \longrightarrow \dot{\xi} = \begin{pmatrix} \dot{\epsilon}_p \\ \dot{\alpha} \end{pmatrix}$$

$$X = -\frac{\partial \Psi}{\partial \xi} \longrightarrow X = -\begin{pmatrix} \frac{\partial \Psi}{\partial \epsilon_p} \\ \frac{\partial \Psi}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \sigma \\ A \end{pmatrix}$$

$$\longrightarrow \begin{cases} \sigma = -\frac{\partial \Psi}{\partial \epsilon_p} & \text{macro stress} \\ A = -\frac{\partial \Psi}{\partial \alpha} & \text{"micro stress"} \end{cases}$$

$$\mathcal{D} = \sigma : \dot{\epsilon} - \dot{\psi} \geq 0$$

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_p \qquad \dot{\psi} = \frac{\partial \psi}{\partial \epsilon_e} : \dot{\epsilon}_e + \frac{\partial \psi}{\partial \alpha} \dot{\alpha}$$

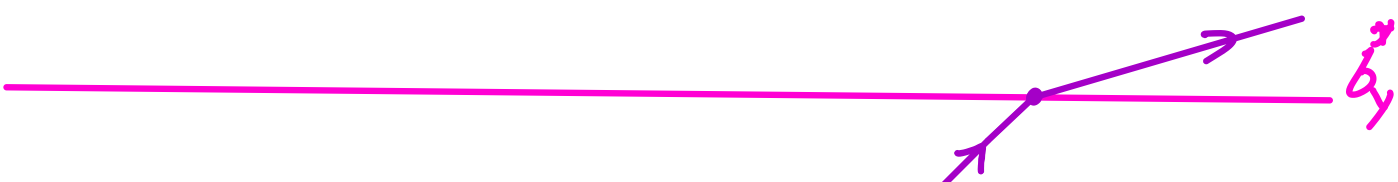
$$\mathcal{D} = \underbrace{\left(\sigma - \frac{\partial \psi}{\partial \epsilon_e} \right)}_{=0} : \dot{\epsilon}_e + \sigma : \dot{\epsilon}_p \overbrace{\boxed{- \frac{\partial \psi}{\partial \alpha}}}^{A} \dot{\alpha} \geq 0$$

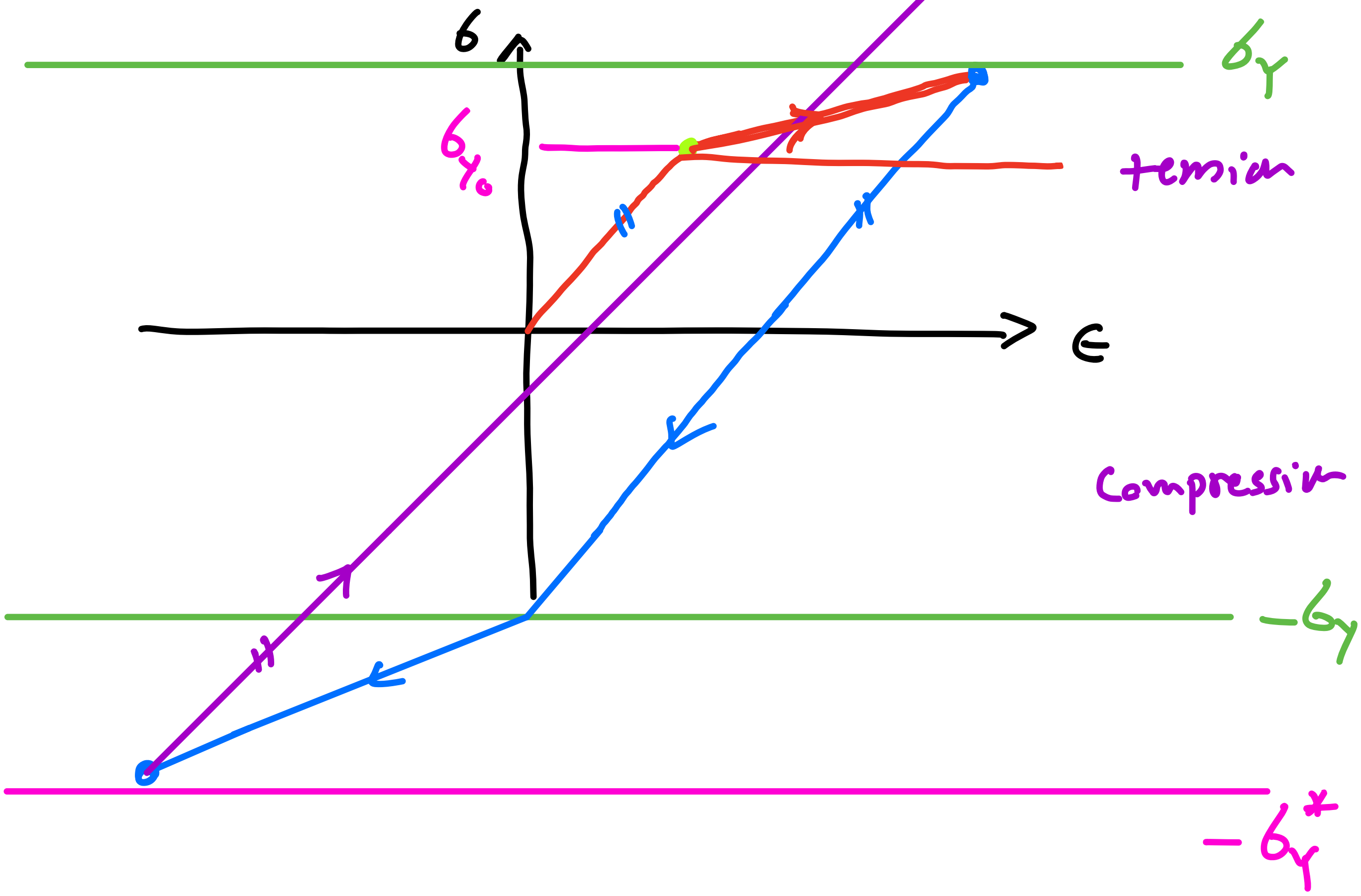
$$\rightarrow \boxed{\sigma = \frac{\partial \psi}{\partial \epsilon_e} = \frac{\partial \psi_e}{\partial \epsilon_e} = \frac{\partial \psi}{\partial \epsilon_p} \underbrace{\frac{\partial \epsilon_p}{\partial \epsilon_e}}_{=-1} = - \frac{\partial \psi}{\partial \epsilon_p}}$$

$$\epsilon_p = \epsilon - \epsilon_e$$

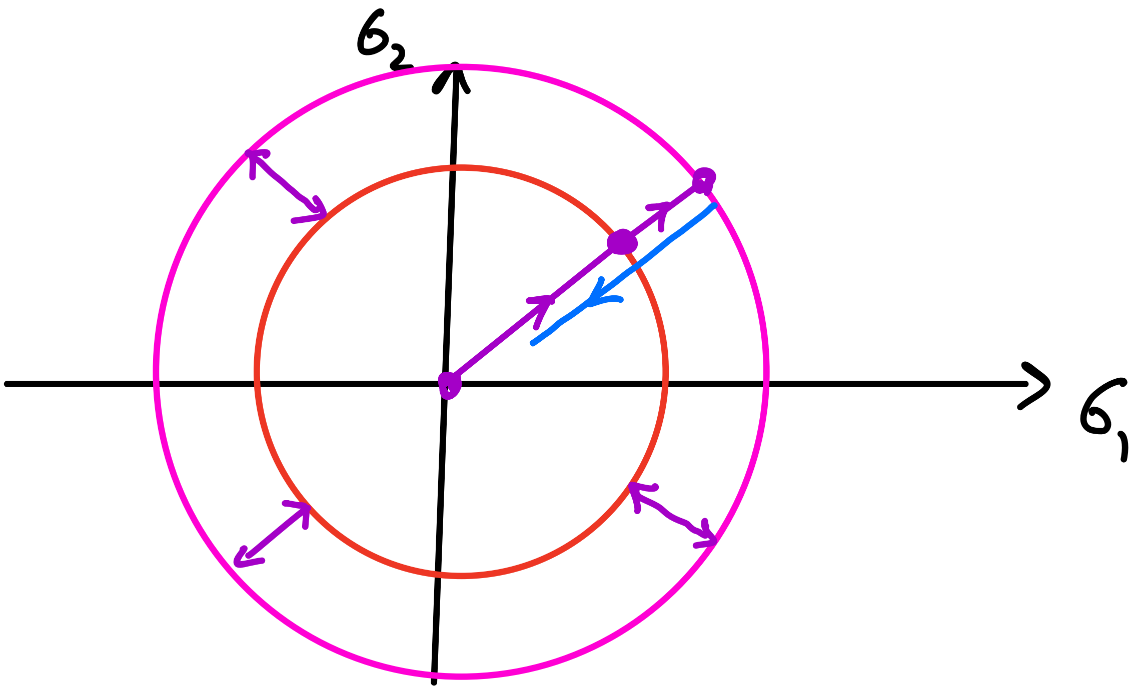
$$\boxed{\mathcal{D} = \underbrace{\sigma}_{F} : \underbrace{\dot{\epsilon}_p}_{J} + \overset{F}{A} \overset{J}{\dot{\alpha}} \geq 0}$$

dissipation integ. of plasticity



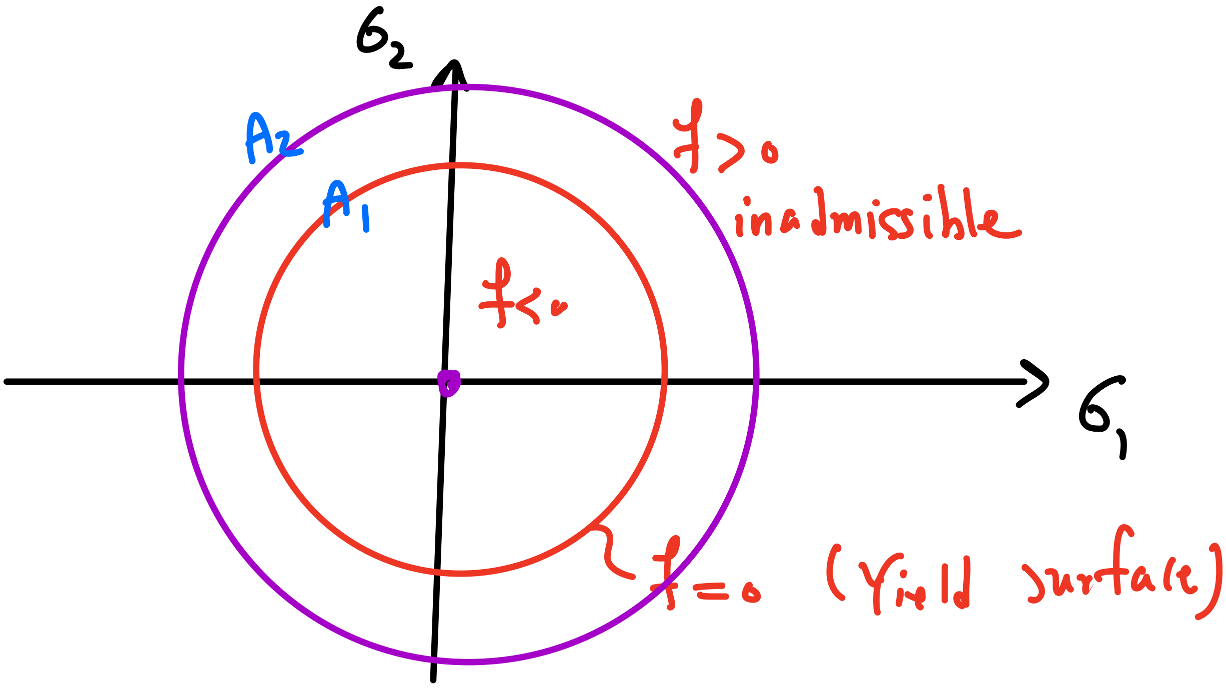


$\sigma \in [-\sigma_Y, \sigma_Y] \equiv E$ elastic region



Def. Yield function and Elastic domain

$f(\sigma, A)$ is called the Yield function:



The elastic domain:

$$E = \{ (\sigma, A) \mid f(\sigma, A) \leq 0 \}$$

<u>Dissipation Potential and its problem</u>

<u>Def.</u> $h(cx) = c^n h(x) \qquad \forall c > 0$

homogeneous of degree $\underline{\underline{n}}$.

<u>prop.</u> In rate-independent plasticity:

$$D = \sigma : \dot{\epsilon}_p + A\dot{\alpha} = \underbrace{\begin{pmatrix} \sigma \\ A \end{pmatrix}}_{\substack{\shortparallel \\ X}} * \underbrace{\begin{pmatrix} \dot{\epsilon}_p \\ \dot{\alpha} \end{pmatrix}}_{\substack{\shortparallel \\ \dot{\xi}}} = X \cdot \dot{\xi} \geq 0$$

The force:

$X = \frac{\partial \Phi}{\partial \dot{\xi}}$

$$X(c\dot{\xi}) = X(\dot{\xi}) \quad \forall c > 0$$

homogeneous of degree zero.

Claim:

$$\Phi(c\dot{\xi}) = c\,\Phi(\dot{\xi})$$

homogeneous of degree one.

$+$ not differentiable at $\dot{\xi} = 0$

Pf. $\boxed{h(c) \equiv \Phi(c\dot{\xi})}$

$$h'(c) = \dot{\xi} \underbrace{\frac{\partial \Phi}{\partial \dot{\xi}}(c\dot{\xi})}_{= X(c\dot{\xi})} = \dot{\xi} \underbrace{X(c\dot{\xi})}_{= X(\dot{\xi})}$$

$$\rightarrow \quad h'(c) = \dot{\xi} X(\dot{\xi}) \xrightarrow{\int dc} \boxed{h(c) = c\,\dot{\xi} X(\dot{\xi})}$$

$h(0) = \Phi(0) = 0$

$$\rightarrow \quad \boxed{\Phi(c\dot{\xi}) = c\dot{\xi}\,X(\dot{\xi}) = c\,\Phi(\dot{\xi})}$$

$c=1$

$$\rightarrow \quad \Phi(\dot{\xi}) = \dot{\xi}\,X(\dot{\xi})$$

Φ is hom. of degree 1

$$\underset{\substack{\| \\ c\dot{\xi}}}{\Phi(y)} = \cancelto{0}{\Phi(0)} + \underbrace{\frac{\partial \Phi}{\partial \dot{\xi}}(0)}_{\substack{\| \\ X_0}}(y-0) + o(y^2)$$

$$\rightarrow \quad \underset{\substack{\| \\ \Phi(\dot{\xi})}}{\boxed{\frac{\Phi(c\dot{\xi})}{c}}} = \cancel{c}\,X_0\cdot\dot{\xi} + \cancelto{0}{\frac{o(c^2)}{c}} \qquad \lim c \to 0$$

$$\rightarrow \quad \Phi(\dot{\xi}) = X_0\cdot\dot{\xi}$$

↯ linearity & $\Phi(\dot{\xi}) \geq 0 \ \forall \dot{\xi}$ incompatible

$\square$

Ex: 1D bar

$\rightarrow \sigma, \epsilon, \epsilon_p$

Assume no hardening var:

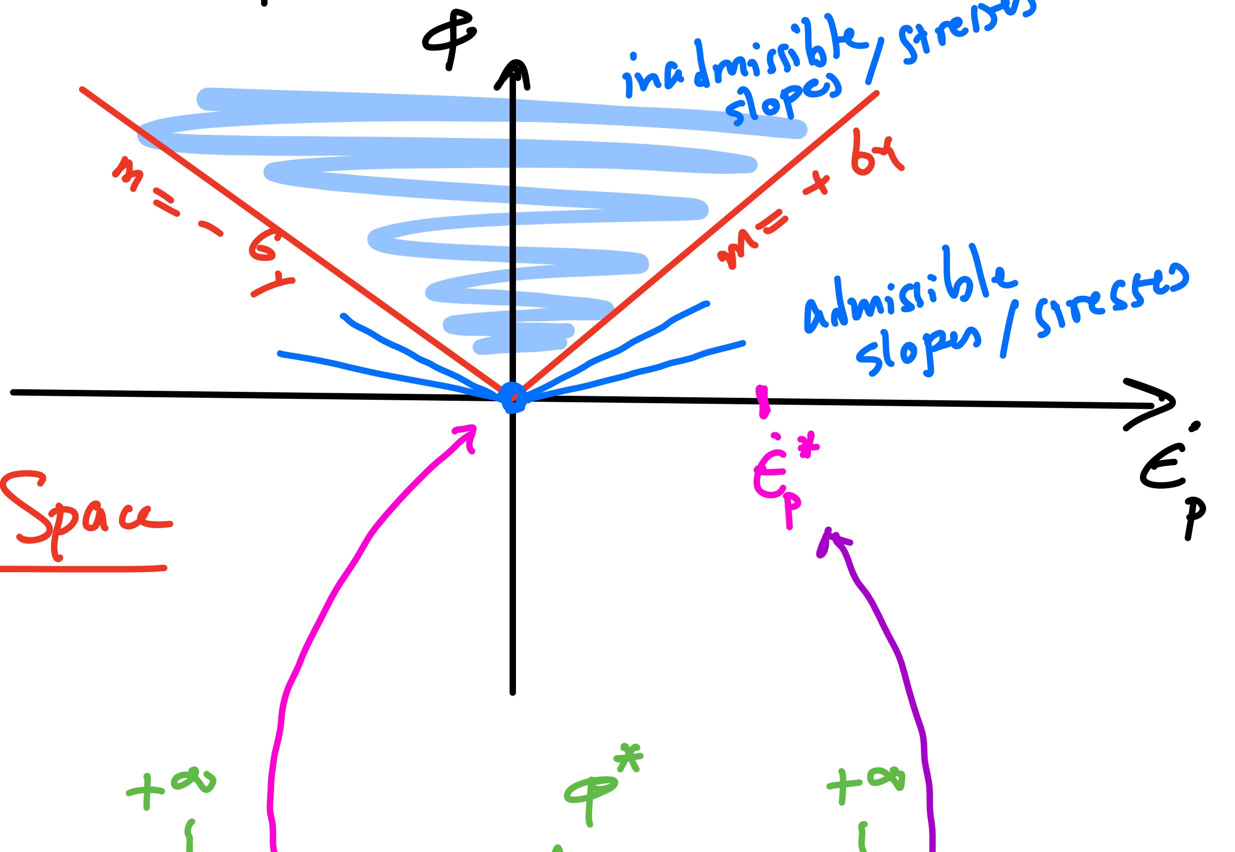
$$\mathcal{D} = \sigma : \dot{\epsilon}_p \geq 0$$

$$\phi(\dot{\epsilon}_p) = \begin{cases} \dot{\epsilon}_p \, \phi(1) & \dot{\epsilon}_p \geq 0 \\ |\dot{\epsilon}_p| \, \phi(-1) & \dot{\epsilon}_p < 0 \end{cases}$$

hom of deg. 1

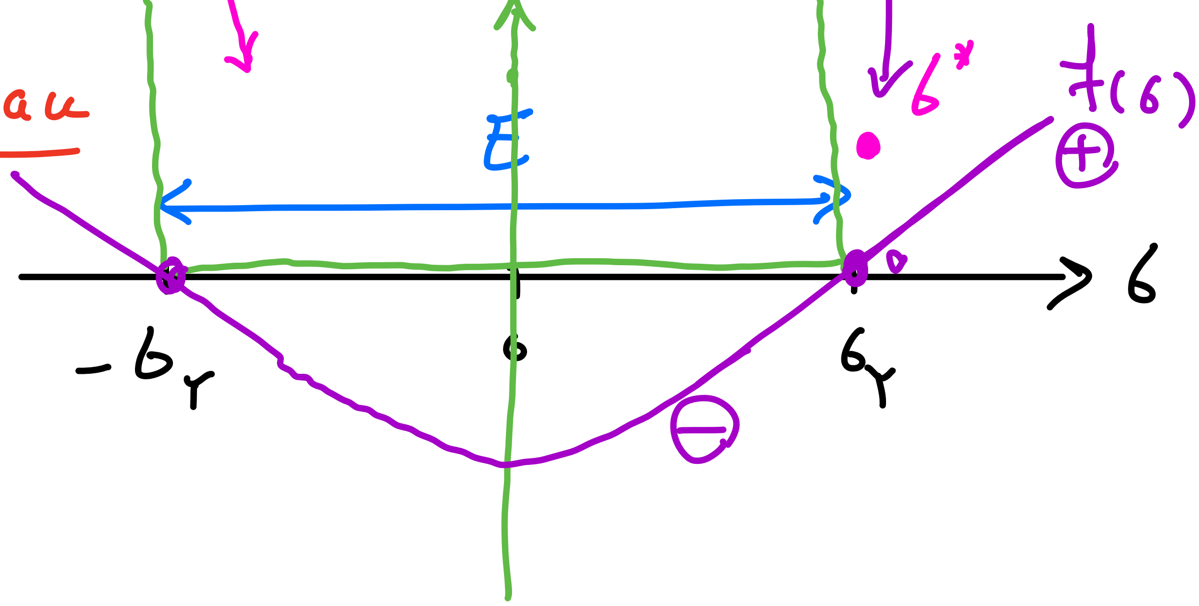
Assume

$\phi(1) = \phi(-1) = \sigma_Y \longrightarrow \boxed{\phi(\dot{\epsilon}_p) = \sigma_Y |\dot{\epsilon}_p|}$



Flux Space

Force Space



$$\varphi^*(\chi) \equiv \max_{\dot{\xi}} \left\{ \chi \cdot \dot{\xi} - \varphi(\dot{\xi}) \right\}$$

$$\dot{\xi} = \frac{\partial \varphi^*}{\partial \chi}$$

<u>Def.</u> Internal variable evolve according to the following <u>flow rules</u>:

mag. of flow

$$\dot{\epsilon}_p = \dot{\gamma} \frac{\partial g}{\partial \sigma} \quad , \quad \dot{\alpha} = \dot{\gamma} \frac{\partial g}{\partial A}$$

plastic multiplier ($\dot{\gamma} \geq 0$)

direction of flow

where $g(\sigma, A)$ is plastic potential.

Flow must be subject to:

$$\dot{\gamma} \geq 0 \quad , \quad f \leq 0 \quad , \quad \dot{\gamma} f = 0$$

Karush – Kuhn – Tucker (KKT) Conditions.

① $\dot{\gamma} = 0 \longrightarrow \dot{\epsilon}_p = 0, \ \dot{\alpha} = 0 \ , \ f < 0$

deformation is elastic

② $\dot{\gamma} > 0 \longrightarrow f = 0$ deformation is plastic

$\dot{\epsilon}_p \neq 0 \ , \ \dot{\alpha} \neq 0$

Def. if $g = f \Longrightarrow$ associative flow rule (metal, —)

if $g \neq f \Longrightarrow$ non_associative " (rocks, soil)

prop. Associative flow satisfies $\mathcal{D} \geq 0$

Assume $f(x)$ is convex

$$\mathcal{D} = \sigma : \dot{\epsilon}_p + A \dot{\alpha} \geq 0 \rightarrow \mathcal{D} = X \cdot \dot{\xi} \geq 0$$

pf. if $\dot{\gamma} = 0 \longrightarrow \dot{\epsilon}_p = 0, \ \dot{\alpha} = 0 \longrightarrow \mathcal{D} = 0$ ✓

$f(0) \leq 0$ if $\dot{\gamma} > 0 \longrightarrow f(X) = 0$

$$\left(\mathcal{D} = \dot{\gamma} \, X \, \frac{\partial f}{\partial X} \geq 0 \quad \text{target} \right)$$

$$\underbrace{\qquad\qquad}_{\geq 0}$$

$$0 \geq f(\cancel{y}_{\to 0}) \geq \underbrace{f(x)}_{\to 0} + \frac{\partial f}{\partial x}(\cancel{y}_{\to 0} - x)$$

$$\longrightarrow \quad x \cdot \frac{\partial f}{\partial x} \geq 0$$

$\square$

prop. flow rule + KKT $\Longrightarrow$ rate independence

p.f. $\hat{\epsilon}_p(t) \equiv \epsilon_p(ct)$ $\quad c > 0$

$$\boxed{\dot{\hat{\epsilon}}_p = c\,\dot{\epsilon}_p(ct) = \underbrace{c\dot{\gamma}}_{=\dot{\hat{\gamma}}} \frac{\partial g}{\partial \sigma}}$$

Thm. Maximum dissipation principle

Given any $\dot{\xi} = (\dot{\epsilon}_p, \dot{\alpha})$ and $X = (\sigma, A)$, the following are equivalent: (admissible)

① $D = X \cdot \dot{\xi} \geq X^* \cdot \dot{\xi} \quad \forall X^*,\ f(X^*) \leq 0$

② flow is associative $(f = g)$:

$$\dot\xi = \dot\gamma \frac{\partial f}{\partial \sigma}, \quad \dot\gamma \geq 0, \quad \dot\gamma f(\chi) = 0$$

pf. in the notes.

<u>Corollary</u>. Maximum Dissipation Principle (MDP)

$$\Longrightarrow \quad \mathcal{D} = \chi \cdot \dot\xi \geq 0$$

<u>pf.</u> $\mathcal{D} = \chi \cdot \dot\xi \geq \cancel{\chi^* \cdot \dot\xi}^{\,0} \quad \forall \chi^*$

pick $\chi^* = 0$ $\square$

<u>Remark</u>. MDP allows defining $\Phi(\dot\xi)$

$$\Phi(\dot\xi) \equiv \max_{f(\chi^*) \leq 0} \chi^* \cdot \dot\xi \underset{\exists \chi}{=} \chi \cdot \dot\xi$$

$$\cancel{\chi = \frac{\partial \Phi}{\partial \dot\xi}} \qquad \chi \in \partial \Phi$$

3. <u>Consistency Condition $\Longrightarrow \dot\gamma$</u>

$$\dot\epsilon_p = \dot\gamma \frac{\partial g}{\partial \sigma}, \quad \dot\alpha = \dot\gamma \frac{\partial g}{\partial A}$$

how to compute ?

During flow $f(x) = 0 \longrightarrow \dot{f} = 0$

$$\dot{\overline{f(\sigma, A)}} = \frac{\partial f}{\partial \sigma} : \dot{\sigma} + \frac{\partial f}{\partial A} \dot{A} = 0$$

✓ ?

$$\dot{\sigma} = \dot{\overline{\mathbb{C} : \epsilon_e}} = \mathbb{C} : \dot{\epsilon}_e$$

$$\sigma = \frac{\partial \psi_e}{\partial \epsilon_e}$$

$$\dot{\epsilon}_e = (\dot{\epsilon} - \dot{\epsilon}_p)$$

$$\dot{\epsilon}_p = \dot{\gamma} \frac{\partial g}{\partial \sigma}$$

$$\dot{\sigma} = \frac{\partial^2 \psi}{\partial \epsilon_e^2} : \dot{\epsilon}_e$$

$$\frac{\partial^2 \psi}{\partial \epsilon_e^2} = \mathbb{C}$$

$$\dot{A} = -\dot{\overline{\frac{\partial \psi_h}{\partial \alpha}}}$$

$$= - \frac{\partial^2 \psi_h}{\partial \alpha^2} \dot{\alpha}$$

$$\dot{\alpha} = \dot{\gamma} \frac{\partial g}{\partial A}$$

$$\rightarrow \frac{\partial f}{\partial \sigma} : \mathbb{C} : \left(\dot{\epsilon} - \dot{\gamma} \frac{\partial g}{\partial \sigma} \right)$$

$$- \frac{\partial f}{\partial A} \frac{\partial^2 \psi_h}{\partial \alpha^2} \frac{\partial g}{\partial A} \dot{\gamma} = 0$$

$$\boxed{\dot{\gamma} = \frac{\frac{\partial f}{\partial \sigma} : \mathbb{C} : \dot{\epsilon}}{\frac{\partial f}{\partial \sigma} : \mathbb{C} : \frac{\partial g}{\partial \sigma} + \underbrace{\frac{\partial f}{\partial A} \frac{d^2 \psi_h}{d\alpha^2} \frac{\partial g}{\partial A}}_{\equiv H \quad \text{hardening modulus}}}}$$

END of Day 3