Variational Calculus (Continued)

4. Functionals w/ higher derivatives

$$J[\omega] = \int_0^L F(x, \omega, \omega', \omega'')\, dx$$

$$\Rightarrow F_\omega - \frac{d}{dx} F_{\omega'} + \frac{d^2}{dx^2} F_{\omega''} = 0 \qquad \text{Euler–Lagrange}$$

on either $x = 0, L$

① $\omega = \bar{\omega}$ or $F_{\omega'} - \frac{d}{dx} F_{\omega''} = 0$

② $\omega' = \bar{\omega}'$ or $F_{\omega''} = 0$

pf.

$$\delta J = \cancel{\int_0^L \overbrace{\left[F_\omega - \frac{d}{dx} F_{\omega'} + \frac{d^2}{dx^2} F_{\omega''} \right]}^{=0} \delta u}$$

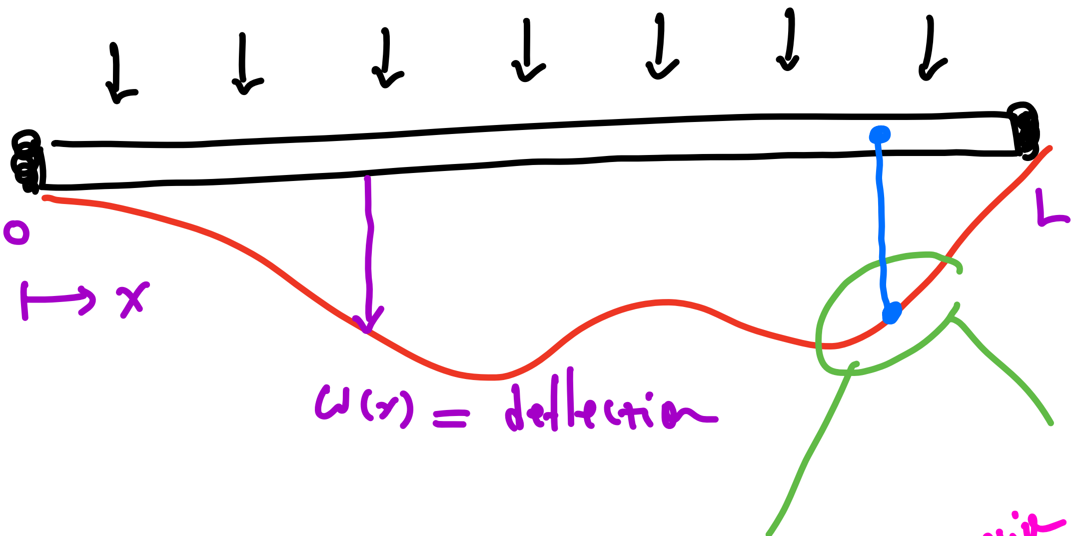
$$+ \left(F_{\omega'} - \frac{d}{dx} F_{\omega''} \right) \delta u \Big|_0^L + F_{\omega''} \delta u' \Big|_0^L = 0$$

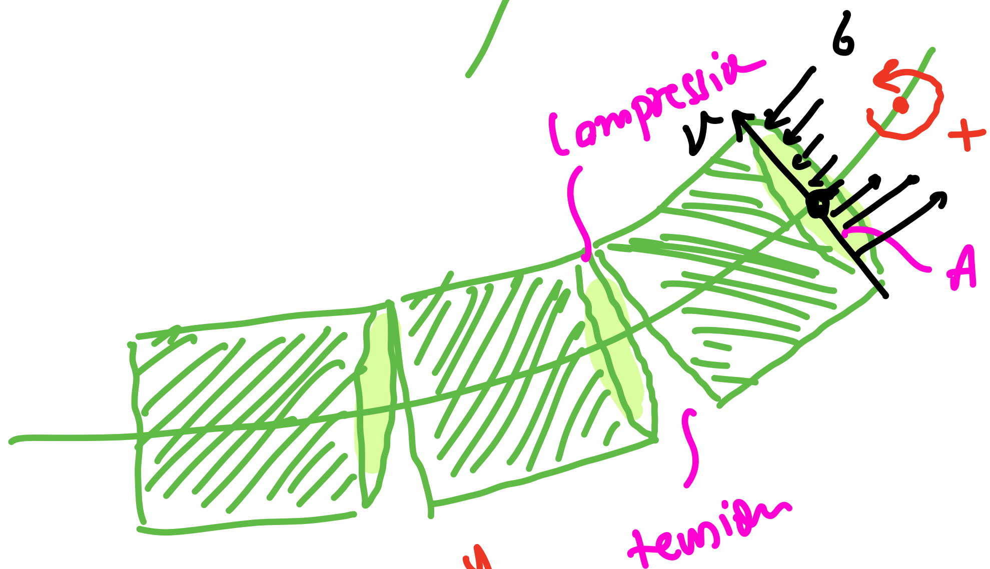
① δu to vanish on $x = 0$ and L

② now pick δu such that it is free at $x = 0, L$

Ex: Euler-Bernoulli Beam Theory

$$q(x) = \frac{\text{Force}}{\text{length}}$$



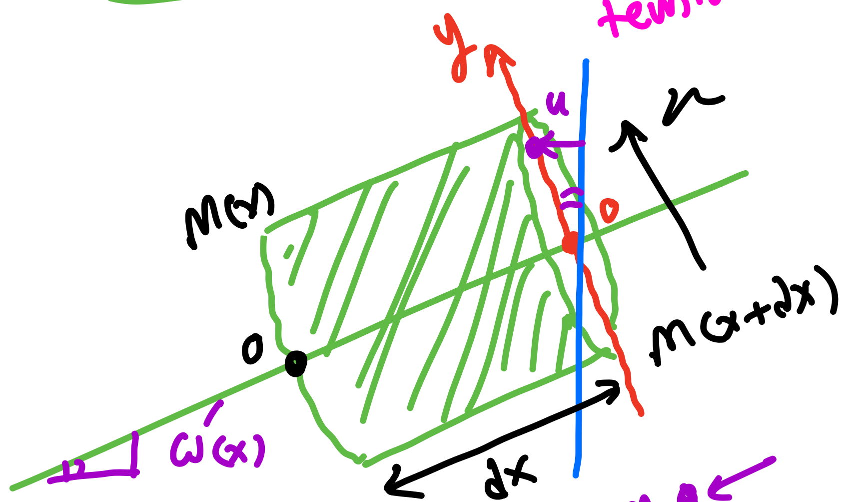


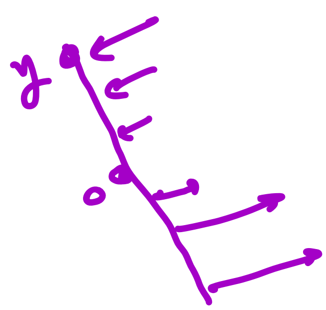
$$\boxed{u(x,y) = -w'y}$$

$$\boxed{\epsilon(x,y) =} \frac{\partial u}{\partial x}$$

$$= \boxed{-yw''}$$

$$\boxed{\sigma(x,y) = E\epsilon(x,y) = -Eyw''}$$





$$\int_A \sigma = -E w'' \int_A y = 0$$

$$\boxed{M} = -\int_A \sigma y = E w'' \int_A y^2 = \boxed{E I w''}$$

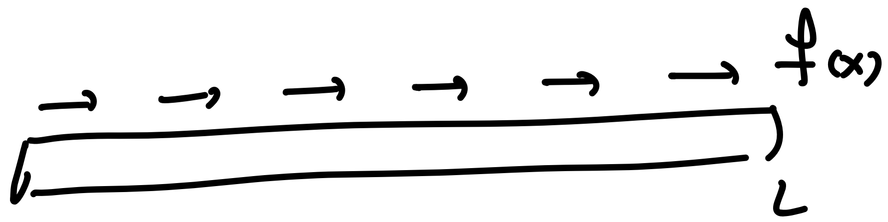
Bending moment

I moment of inertia

$$V = ? \longrightarrow \underbrace{M(x+dx) - M(x)}_{= dM} + V\,dx = 0$$

$$\longrightarrow V = -\frac{dM}{dx} \longrightarrow \boxed{V = -M'}$$

<u>Recall</u>: For a bar under tension



$$\Pi[u] = \int_0^L \Big[\underbrace{\tfrac{1}{2} E A \epsilon^2 - f u}_{F(u,u')}\Big] dx \longrightarrow \delta\Pi = 0$$

$\epsilon = u'$

$$\longrightarrow (E A u')' + f = 0$$

$$\boxed{\Pi[w]} = \int_0^L \iint_A \frac{1}{2} E \underset{\uparrow \text{ Subst.}}{\epsilon^2} - \int_0^L q w = \boxed{\int_0^L \underbrace{\left(\frac{1}{2} EI w''^2 - q w\right)}_{F(w,\, w'')} dx}$$

$$\delta \Pi(w) = \int_0^L \underbrace{\left(EI w'''' - q\right)}_{=0} \delta w \, dx + EI w'' \delta w' \Big|_0^L - EI w''' \delta w \Big|_0^L = 0$$

E-L : $(EI w'')'' = q$

Compatible BCs

① Clamped : $w(0) = 0$, $w'(0) = 0$

② Simply supported : $w(0) = 0$, $M = 0 \Rightarrow w''(0) = 0$

③ Fully free : $M = V = 0$

$\longrightarrow w''(0) = 0$, $w'''(0) = 0$

5. Higher-order variation, minimums

Def. 2nd variation

$$\delta^2 J \equiv \frac{d^2}{d\epsilon^2} J[u+\epsilon\eta] \Big|_{\epsilon=0}$$

Note: Taylor Expand in ϵ

$$J[u+\epsilon\eta] = J[u] + \delta J\,\epsilon + \delta^2 J\,\frac{\epsilon^2}{2} + O(\epsilon^3)$$

Note: Can compute $\delta^2 J$ by applying the δ-rule to δJ.

Ex: $J[u] = \int_0^L F(x,u,u')\,dx$

$$\xrightarrow{\delta} \quad \delta J = \int_0^L \delta F\,dx$$

$$= \int_0^L (F_u\,\delta u + F_{u'}\,\delta u')\,dx$$

$$\xrightarrow{\delta} \delta^2 J = \int_0^L F_{uu}\,\delta u^2 + \cancel{F_{uu'}\,\delta u\,\delta u'} + \cancel{F_{u'u}\,\delta u'\,\delta u} + F_{u'u'}\,\delta u'^2$$

$$2F_{uu'}\,\delta u\,\delta u'$$

Ex: Consider 1D bar under tension

$$\Pi[u] = \int_0^L \left(\frac{1}{2} EA u'^2 - f u\right) dx$$

$$\xrightarrow{\delta} \boxed{\delta\Pi = \int_0^L EA\,u'\,\delta u' - f\,\delta u} = 0 \qquad \delta\delta u' = 0$$

$$\xrightarrow{\delta} \delta^2\Pi = \int_0^L EA\,\underbrace{\delta u'^2}_{=\,\delta u'\,\delta u'} \geq 0$$

$$\xrightarrow{\delta} \delta^3\Pi = 0$$

$$\longrightarrow \Pi[\underline{\underline{u}} + \epsilon\eta] - \Pi[u] + \cancelto{0}{\delta\Pi\,\epsilon} = \delta^2\Pi\,\frac{\epsilon^2}{2} > 0$$

if u satisfies $\delta\Pi = 0$

$$\rightarrow \boxed{\Pi[u+\epsilon\eta] \geq \Pi[u]}$$

Satisfies $\delta\Pi = 0$

Remark: Weak form

$$\int_0^L EA\,u'\,\delta u' - f\,\delta u = 0$$

trial func. $\equiv$ lower regularity demanded

test func.

Virtual Work and Energy Thms

1. Two notions of "work", and strain Energy

Def. Thermodynamic work

1st law

$$U = W_{act} + Q$$

internal Energy

work done

heat

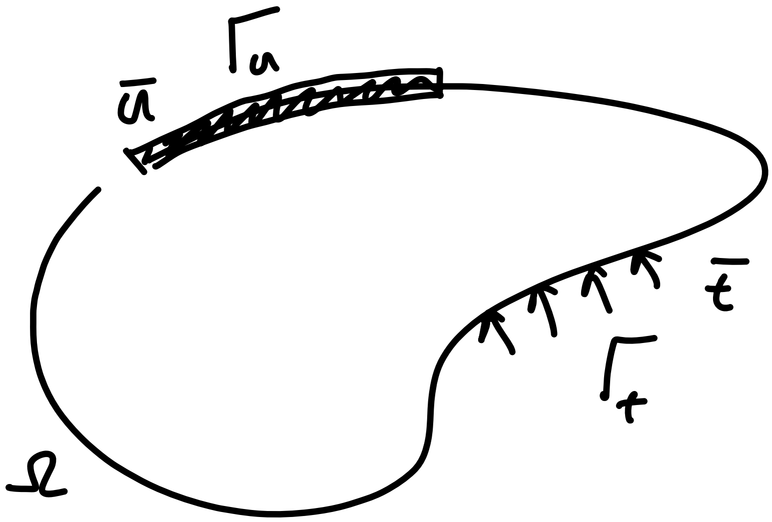
$$W_{act} = \int f \cdot du$$

loading path

Def. Virtual work (or load functional)

$$W_{ext}[u] = \int_{\Omega} b \cdot u \, dV + \int_{\Gamma_t} \bar{t} \cdot u \, dA$$

$\forall u$ admissible "u"
($u=0$ on Γ_u)



$$\partial\Omega = \Gamma_u \cup \Gamma_t$$

Def. Strain Energy functional

hyperelastic material, there exists ψ s.t.

$$\sigma = \frac{\partial \psi}{\partial \epsilon}, \quad \psi = \frac{\text{energy stored}}{\text{bulk vol.}}$$

$$U[u] = \int_{\Omega} \psi(\epsilon) \, dV$$ total Energy stored (elastically)

2. Total potential, Priciple of Virtual work

$\delta\Pi = 0$

Def.

$$\Pi[u] = U[u] - W_{ext}[u]$$

Thm. Principle of Virtual Work (PVW)

$$\delta\Pi = 0 \iff \text{weak form} \iff \text{strong form}$$

$$\boxed{\int_\Omega \sigma : \delta\epsilon - \int_\Omega b \cdot \delta u - \int_{\Gamma_t} \bar{t} \cdot \delta u = 0}$$

solved in finite elements.

$$\boxed{\begin{aligned} \nabla \cdot \sigma + b &= 0 \\ \sigma n = \bar{t} \quad &\text{on } \Gamma_t \end{aligned}}$$

linear mom. Balance

pf.

$$\sigma = \frac{\partial \Psi}{\partial \epsilon}$$

$$\frac{\partial \Psi}{\partial \epsilon} : \delta\epsilon = \delta\Psi(\epsilon)$$

$$\delta\Pi = \int_\Omega \delta\Psi(\epsilon) - \int_\Omega b \cdot \delta u - \int_{\Gamma_t} \bar{t} \cdot \delta u$$

$$\delta \longrightarrow \quad \delta\Pi = \int_{\Omega} \overset{\text{sym}}{\underbrace{\sigma : \delta\epsilon}} - \int_{\Omega} b \cdot \delta u - \int_{\Gamma_t} \bar{t} \cdot \delta u = 0$$

$$\delta\epsilon = \delta \nabla_s u = \nabla_s \delta u$$

$$\downarrow$$

sym.

(Weak $\Rightarrow$ Strong)

$$\boxed{\sigma : \delta\epsilon = \sigma : \nabla \delta u}$$

$$\underbrace{\int \sigma : \nabla \delta u}_{=}$$

$$\underbrace{\int_{\Omega} \nabla \cdot (\sigma \delta u)}_{=} - \int_{\Omega} (\nabla \cdot \sigma) \delta u$$

$$\int_{\cancel{\partial\Omega}\ \Gamma_t} \sigma n \cdot \delta u$$

collecting $\longrightarrow$

$$-\int_{\Omega} \underbrace{(\nabla \cdot \sigma + b)}_{=0} \underline{\delta u} + \int_{\Gamma_t} \underbrace{(\sigma n - \bar{t})}_{=0} \delta u = 0$$

$\square$

Thm: Clapyron's Thm

$$U = W_{act} = \frac{1}{2} W_{ext}$$

(assuming $Q=0$)

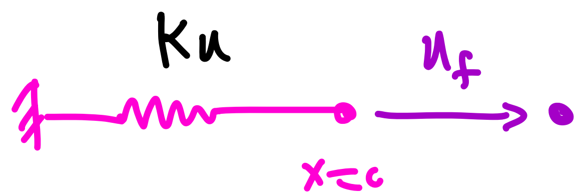
Caveat

if $u=0$ at Γ_u

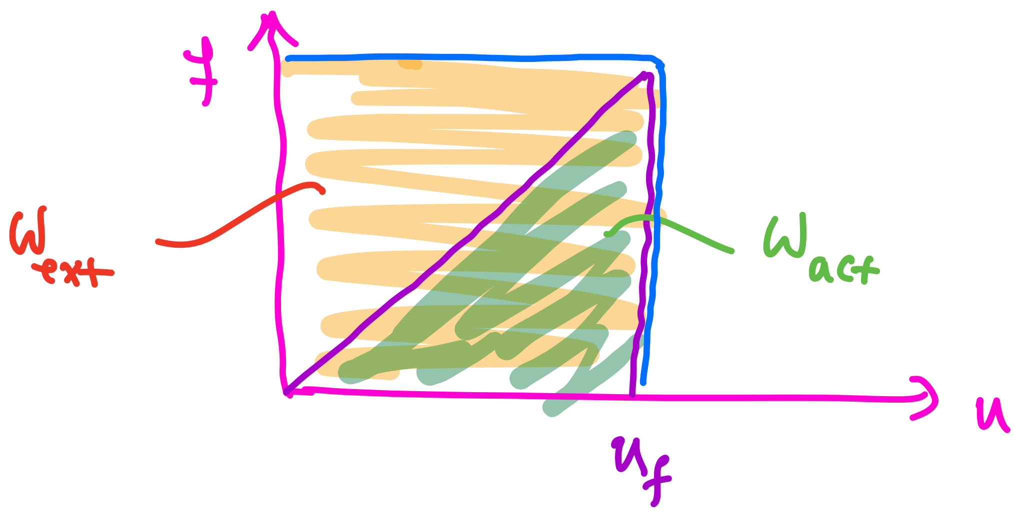
+

linear Elastic Body

Ex:



$$\begin{cases} W_{act} = \int_0^{u_f} Ku\,du = \frac{1}{2} Ku^2 \\ W_{ext} = f \cdot u = Ku^2 \end{cases} \longrightarrow W_{act} = \frac{1}{2} W_{ext}$$



pf. From weak form:

$$\int_\Omega \sigma : \delta\epsilon = W_{ext}[\delta u]$$

$\delta\epsilon \downarrow \epsilon$

$\delta u \downarrow u$ solution E-L

admissible

$$\longrightarrow \quad \int_\Omega \overset{\mathbb{C}:\epsilon}{\overset{\shortparallel}{\sigma}} : \epsilon = W_{ext} \qquad\qquad \underline{\sigma = \mathbb{C}:\epsilon}$$

linear

$$\shortparallel$$

$$\int_\Omega \epsilon : \mathbb{C} : \epsilon = 2 \int_\Omega \psi = 2U \underset{1^{st}\text{ law}}{=} 2W_{act} \qquad \square$$

Thm. $\delta \Pi = 0 \Longrightarrow u$ s.t. it minimizes $\Pi[u]$

provided $\mathbb{C}$ is positive definite

$$\underbrace{\delta\epsilon : \frac{\partial \sigma}{\partial \epsilon}}_{\delta\sigma : \delta\epsilon} \qquad (\epsilon : \mathbb{C} : \epsilon > 0, \ \forall \epsilon)$$

Pf.

$$\delta^2 \Pi = \int_\Omega \delta^2\psi - \cancel{\delta^2 \int_\Omega b \cdot u}^{\,0} - \cancel{\delta^2 \int_{\Gamma_t} \bar{t} \cdot u}^{\,0}$$

$$\longrightarrow \quad \delta^2 \Pi = \int_\Omega \overbrace{\delta\epsilon : \underbrace{\frac{\partial \sigma}{\partial \epsilon}}_{\mathbb{C}} : \delta\epsilon}^{>0} > 0 \qquad \sigma = \mathbb{C} : \epsilon$$

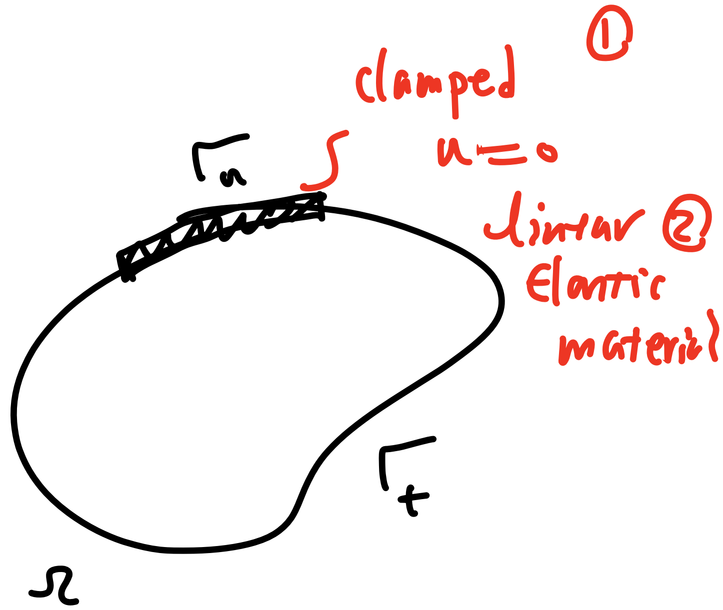
$\longrightarrow$ u minimizes Π $\square$

3. Reciprocity & Castigliano's thms

Thm. Betti's reciprocity

① $(\bar{t}_1, b_1) \longrightarrow u_1(x)$

② $(\bar{t}_2, b_2) \longrightarrow u_2(x)$



$$W^1_{ext}[u_2] = W^2_{ext}[u_1]$$

$$\longrightarrow \int_\Omega b_1 \cdot u_2 + \int_{\Gamma_t} \bar{t}_1 \cdot u_2 = \int_\Omega b_2 \cdot u_1 + \int_{\Gamma_t} \bar{t}_2 \cdot u_1$$

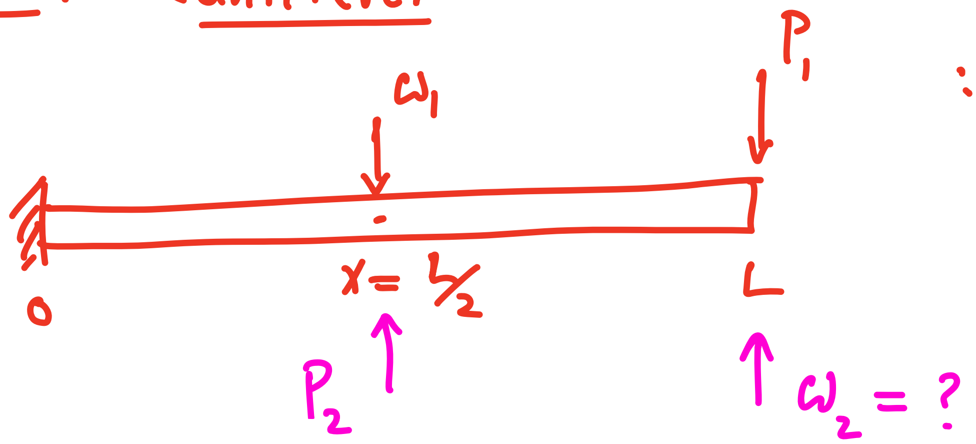
pf. u_1 & u_2 admissible variations

$$\int_\Omega \sigma^i : \delta\epsilon = W^i_{ext}[\delta u] \qquad i = 1, 2$$

$$\int \sigma^1 : \epsilon_2 = \int \epsilon_1 : \mathbb{C} : \epsilon_2 = \int \sigma^2 : \epsilon_1 \qquad \square$$

$\mathbb{C} : \epsilon_1$

Ex: Cantilever



: System 1

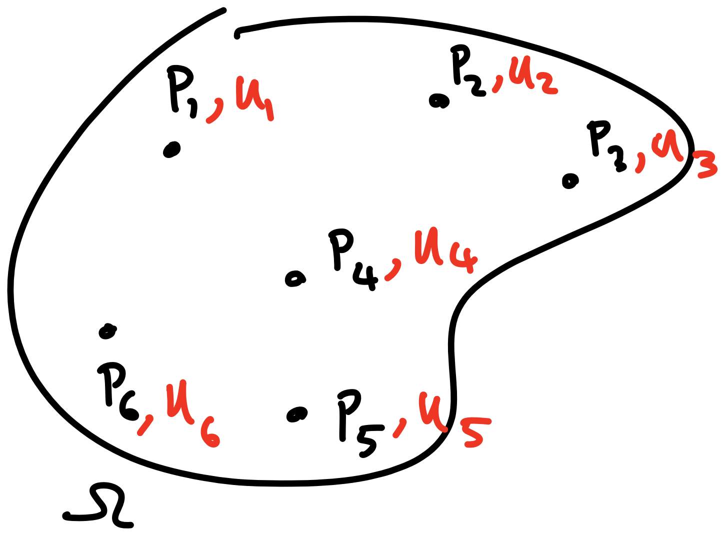
: System 2

Betti says: $P_1 w_2 = P_2 w_1 \longrightarrow w_2 = \dfrac{P_2 w_1}{P_1}$

Point-load systems:

$$U = U(u_1, u_2, \ldots, u_n)$$

$$W_{ext} = \sum_i P_i u_i$$



$$\Pi(u_1, u_2, \ldots, u_n) = U - \sum_i P_i u_i$$

$$\nabla \cancel{\delta} \Pi = 0 \longrightarrow \frac{\partial \Pi}{\partial u_i} = 0$$

$$\longrightarrow \boxed{\frac{\partial U}{\partial u_i} = P_i} \quad \text{I}$$

What if $\boxed{u_i \doteq \frac{\partial U^*}{\partial P_i}}$ II

$$U^*(P_1, P_2, \longrightarrow, P_n) \equiv \sum_j P_j u_j - U(u_1, \longrightarrow, u_n)$$

$$\longrightarrow \boxed{\frac{\partial U^*}{\partial P_i} = u_i} + \cancel{\sum_j P_j \frac{\partial u_j}{\partial P_i}} - \cancel{\sum_j \frac{\partial U}{\partial u_j} \frac{\partial u_j}{\partial P_i}}$$

P_j
$=$

Def. linear structure

$$\vec{P} = K\vec{U} \longrightarrow P_i = \sum_j K_{ij} u_j$$

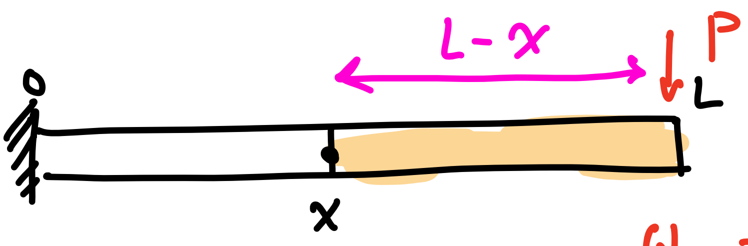
prop. $\boxed{U = \frac{1}{2}\sum_{i,j} u_i K_{ij} u_j = \frac{1}{2}\sum_i P_i u_i}$

$$\boxed{U^* = U}$$

$$U^* = \sum_i P_i u_i - U = \frac{1}{2}\sum_i P_i u_i = U$$

$$P_i = \frac{\partial U}{\partial u_i}, \quad u_i = \frac{\partial U^*}{\partial P_i} = \frac{\partial U}{\partial P_i}$$

Ex: Beam / Cantilever tip deflection



$w_{tip} = ?$

$$M(x) = P(L-x)$$

$$\parallel$$

$$EI\,w''$$

$$\longrightarrow \boxed{w'' = \frac{P}{EI}(L-x)}$$

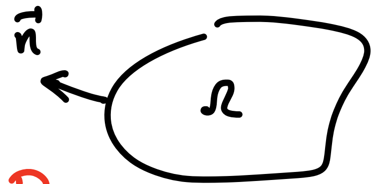
$$\longrightarrow \boxed{w(x) = \frac{Px^2(3L-x)}{6EI}}$$

$$\boxed{U = \int_0^L \frac{1}{2} EI\, w''^2} = \boxed{\frac{P^2L^3}{6EI}}$$

$$\boxed{u_{tip} = \frac{\partial U^*}{\partial P} = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI}}$$

Thermodynamics & Dissipation Potentials

1. Two laws of thermo (in power), dissipation inequality



Axiom. 1st law

$\cancel{P_{ext}}$ P_{int}

heat flux

$$\frac{d}{dt}\int_\Omega \dot{e}\, dV = \boxed{\cancel{P_{act}}} - \int_{\partial\Omega} q \cdot n\, dA + \int_\Omega r\, dV$$

internal energy / bulk vol.

heat rate / vol.

$$P_{act} = P_{ext} \qquad (W_{act} \neq W_{ext})$$

$$W_{act} = \int_0^t \int_\Omega f \cdot \dot{u}\, dt \longrightarrow \dot{W}_{act} = \int_\Omega f \cdot \dot{u} \equiv P_{ext}$$

$$P_{ext} = \int_\Omega b\cdot\dot u + \int_{\Gamma_t} \bar t\cdot\dot u \underset{\text{weak form}}{=\!=} \int_\Omega \sigma:\dot\epsilon = P_{int}$$

Prop. Local form of 1st-law

$$\boxed{\dot e = \sigma:\dot\epsilon - \nabla\cdot q + r}$$

Pf.

$$\int_\Omega \big(\dot e \underbrace{- \sigma:\dot\epsilon + \nabla\cdot q - r}_{=0}\big)\, dV = 0$$

$\forall \Omega$

$\square$

Axiom: 2nd law of thermo

$$\dot s = -\nabla\cdot\left(\frac{q}{T}\right) + \frac{r}{T} + \underbrace{\dot s_G}_{\geq 0}$$

$$\rightarrow \boxed{\dot s \geq -\nabla\cdot\left(\frac{q}{T}\right) + \frac{r}{T}}$$

Def. Helmholtz free energy

$$\underbrace{\psi}_{\text{helmholtz}} \equiv e - TS \qquad \text{legendre transf.}$$

Thm. Clausius – Duhem inequality

$$\boxed{\mathcal{D} \equiv \sigma : \dot{\epsilon} - \dot{\psi} - s\dot{T} - \frac{q \cdot \nabla T}{T} \geq 0}$$

Pf.

$$T\left(\dot{s} \geq \underbrace{-\nabla \cdot \left(\frac{q}{T} \right)}_{= -\frac{\nabla \cdot q}{T} + \frac{\nabla T \cdot q}{T^2}} \right)$$

lets ignore "r" for simplicity

$$\rightarrow \quad T\dot{s} \geq -\underbrace{\nabla \cdot q}_{= \sigma : \dot{\epsilon} - \dot{e}} + \frac{q \cdot \nabla T}{T}$$

$$\dot{\psi} - T\dot{s} - s\dot{T}$$

$$\rightarrow \cancel{T\dot{s}} \geq \sigma : \dot{\epsilon} - \dot{\psi} + \cancel{T\dot{s}} + \underline{s\dot{T}} + \frac{q \cdot \nabla T}{T}$$

$$\rightarrow \sigma : \dot{\epsilon} - \dot{\psi} - s\dot{T} - \frac{q \cdot \nabla T}{T} \geq 0$$

$\square$

Corollary: Constant & uniform ($T = ct$)

$$\boxed{\mathcal{D} = \sigma : \dot{\epsilon} - \dot{\psi} \geq 0}$$

Remark: For hyperelastic material $\underline{\psi = \psi(\epsilon)}$

$\sigma = \sigma(\epsilon)$

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon} : \dot{\epsilon}$$

$(\dot{T}, \dot{\epsilon})$ are externally controlable variables

$$\rightarrow \mathcal{D} = \underbrace{\overbrace{\left(\sigma_{(\epsilon)} - \frac{\partial \psi}{\partial \epsilon} \right)}^{= 0}}_{\neq f(\dot{\epsilon})} : \underbrace{\dot{\epsilon}}_{\text{arbitrary}} \geq 0$$

$$\rightarrow \boxed{\sigma = \frac{\partial \psi}{\partial \epsilon}} \rightarrow \boxed{D = 0}$$

2. Coleman-Noll procedure

Thermo-elastic medium

$$\begin{cases} \psi = \psi(\epsilon, T) \\ \sigma = \sigma(\epsilon, T) \\ s = s(\epsilon, T) \\ q = q(\epsilon, T, \nabla T) \\ \dot{\epsilon}, \dot{T} = \text{externally controllable} \end{cases}$$

$$D = \sigma : \dot{\epsilon} - \underline{\underline{\dot{\psi}}} - s\dot{T} - \frac{q \cdot \nabla T}{T} \geq 0$$

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon} : \dot{\epsilon} + \frac{\partial \psi}{\partial T} \dot{T}$$

$$\rightarrow D = \underbrace{\left(\sigma - \frac{\partial \psi}{\partial \epsilon}\right)}_{\neq f(\dot{\epsilon})} : \overset{\text{arbitrary}}{\dot{\epsilon}} - \overbrace{\left(s + \frac{\partial \psi}{\partial T}\right)}^{\neq f(\dot{T})} \dot{T} - \frac{q \cdot \nabla T}{T} \geq 0$$

arbitrary

$$\rightarrow \boxed{\begin{gathered} \mathcal{D} = -\frac{q \cdot \nabla T}{T} \geq 0 \\ \sigma = \frac{\partial \psi}{\partial \epsilon}, \quad s = -\frac{\partial \psi}{\partial T} \end{gathered}}$$

Corollary: Maxwells relation

$$\frac{\partial^2 \psi}{\partial \epsilon \partial T} = \frac{\partial^2 \psi}{\partial T \partial \epsilon} \longrightarrow \boxed{\frac{\partial \sigma}{\partial T} = -\frac{\partial s}{\partial \epsilon}}$$

Strain-rate dependent processes ($T = \text{ct}$)

$$\psi = \psi(\epsilon, \dot{\epsilon})$$

$$\sigma = \sigma(\epsilon, \dot{\epsilon})$$

dissipation

$$\mathcal{D} = \sigma : \dot{\epsilon} - \underline{\dot{\psi}} \geq 0$$

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon} : \dot{\epsilon} + \frac{\partial \psi}{\partial \dot{\epsilon}} : \ddot{\epsilon}$$

σ_v viscous stress

arbitrary

$$\rightarrow \quad \mathcal{D} = \underbrace{\overbrace{\left(\sigma - \frac{\partial \psi}{\partial \epsilon}\right)}^{\sigma_v}}_{= f(\dot{\epsilon}, \epsilon)} : \dot{\epsilon} - \underbrace{\overbrace{\frac{\partial \psi}{\partial \dot{\epsilon}}}^{0}}_{\neq f(\ddot{\epsilon})} : \ddot{\epsilon} \geq 0$$

arbitrary

Controllable

$$\rightarrow \quad \boxed{\begin{aligned} &\mathcal{D} = \sigma_v : \dot{\epsilon} \geq 0 \\ &\frac{\partial \psi}{\partial \dot{\epsilon}} = 0 \quad , \quad \sigma = \frac{\partial \psi}{\partial \epsilon} + \sigma_v \end{aligned}}$$

$(T = ct)$

3. Internal Variables, dissipation Potential

$\psi = \psi(\epsilon, \xi)$ internal variable(s)

$\sigma = \sigma(\epsilon, \xi)$

$\xi, \dot{\xi}, \dots =$ Cannot be externally controlled

$\xi =$ captures internal changes

(yield, fracture, dislocation / grain rearrangements)

Prop. $\psi = \psi(\epsilon, \dot{\epsilon}, \xi)$

$$\sigma = \sigma(\epsilon, \dot{\epsilon}, \xi)$$

$$\mathcal{D} = \sigma : \dot{\epsilon} - \dot{\psi} \geq 0$$

$$\dot{\psi} = \frac{\partial \psi}{\partial \epsilon} : \dot{\epsilon} + \frac{\partial \psi}{\partial \dot{\epsilon}} : \ddot{\epsilon} + \frac{\partial \psi}{\partial \xi} \dot{\xi}$$

$$\rightarrow \quad \mathcal{D} = \overbrace{\underbrace{\left(\sigma - \frac{\partial \psi}{\partial \epsilon}\right)}_{= f(\dot{\epsilon})}}^{\sigma_r} : \dot{\epsilon} - \underbrace{\frac{\partial \psi}{\partial \dot{\epsilon}}}_{0} : \ddot{\epsilon} - \underbrace{\frac{\partial \psi}{\partial \xi}}_{X \text{ "conjugate force"}} \overbrace{\dot{\xi}}^{\text{"flux"}} \geq 0$$

$$\rightarrow \quad \boxed{\begin{array}{l} \mathcal{D} = \sigma_r : \dot{\epsilon} + X \dot{\xi} \geq 0 \\[2ex] \dfrac{\partial \psi}{\partial \dot{\epsilon}} = 0, \quad \sigma = \dfrac{\partial \psi}{\partial \epsilon} + \sigma_r \end{array}}$$

Def. often terms like $F.J$ appear in $\mathcal{D}$.

$$\begin{cases} F = \text{Conjugate force} \\ J = \text{flux} \end{cases}$$

$$\underset{F}{\sigma_v} \;,\; \underset{J}{\epsilon} \qquad\qquad \underset{F}{X} \;,\; \underset{J}{\dot{\xi}}$$

(σ_v, ϵ), $\sigma_v = F$, $\epsilon = J$

$(X, \dot{\xi})$, $X = F$, $\dot{\xi} = J$

A <u>dissipation potential</u> is a function defined/corresponding to the pair (F, J)

$$\Phi(J) \geq 0, \quad \Phi(0) = 0, \quad \text{Convex}, \quad F = \frac{\partial \Phi}{\partial J}$$

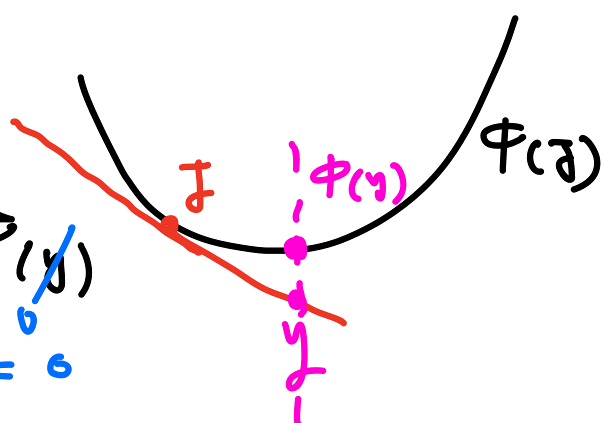
dissipation potential

<u>Thm</u>. for a dissip. potential $\Phi(J)$

$$F \cdot J = \frac{\partial \Phi}{\partial J} \cdot J \geq \Phi(J) \geq 0$$

Pf.

$$\Phi(J) + \frac{\partial \Phi}{\partial J}(\cancelto{0}{y} - J) \leq \Phi(\cancelto{0}{y}) = 0$$



Pick $y = 6$

END of Day 2