Continuum mechanics & fundamentals

1. Tensors & index notation

$a \cdot b = a_i b_i$ rule = repeated indices sum

vectors

$= a_1 b_1 + a_2 b_2 + a_3 b_3 \quad i \in \{1, 2, 3\}$

$A : B = A_{ij} B_{ij}$

tensor

$(\sigma n)_i = \sigma_{ij} n_j$

tensor

vector

free index

$ijk \quad jki \quad kij$

$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Kronecker delta

$\epsilon_{ijk} = \begin{cases} 1 & \text{even perm.} \\ 0 & \text{repeated} \\ -1 & \text{odd perm} \end{cases}$

$e_{112} = 0$

$\delta_{ij} a_j = a_i$

$(a \times b)_i = \epsilon_{ijk} a_j b_k$

Cross product

free

$$(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j} = u_{i,j}$$

$$\nabla \cdot u = \frac{\partial u_i}{\partial x_i} = u_{i,i}$$

$$(\nabla \cdot \sigma)_i = \sigma_{ij,j}$$

2nd order tensor

$$A = A^T \quad \text{symmetric} \quad A_{ij} = A_{ji}$$

$$-A = A^T \quad \text{antisymmetric} \quad A_{ij} = -A_{ji}$$

$$A = \underbrace{\frac{A + A^T}{2}}_{A_s} + \underbrace{\frac{A - A^T}{2}}_{A_a}$$

Proposition: $S = S^T, \; A = -A^T \Longrightarrow A : S = 0$

pf.

$$A : S = A_{ij} S_{ij} = \underset{-A_{ij}}{A_{ji}} \, \underset{S_{ij}}{S_{ji}} = -A_{ij} S_{ij}$$

$||$

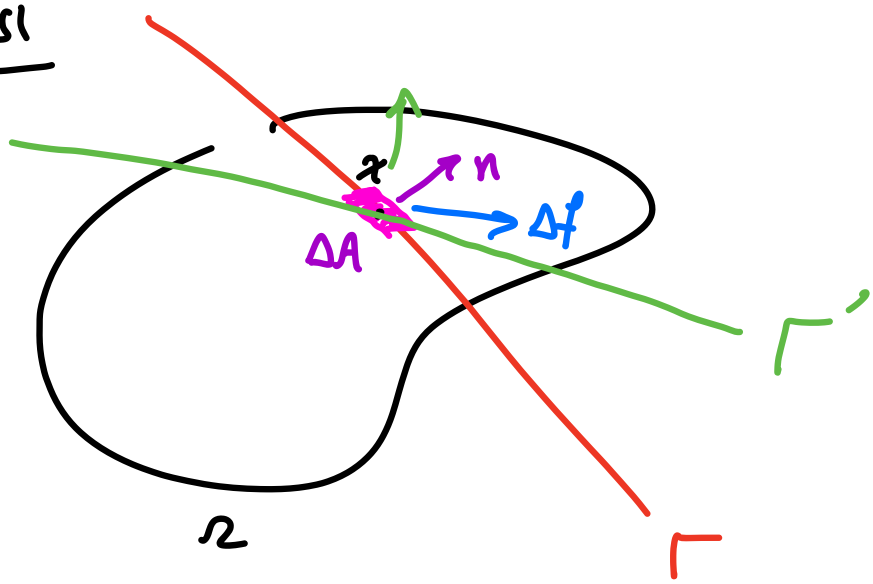
$$A_{ij}\delta_{ij} = 0$$

□

<u>Corollary</u>: $S = S^T$, A any 2^{nd}-order tensor

$$\Rightarrow S:A = S:A_s + \underbrace{S:A_a}_{=0}$$

2. Traction & Stress



$$t(x,n) = \lim_{\Delta A \to 0} \frac{\Delta f}{\Delta A}$$

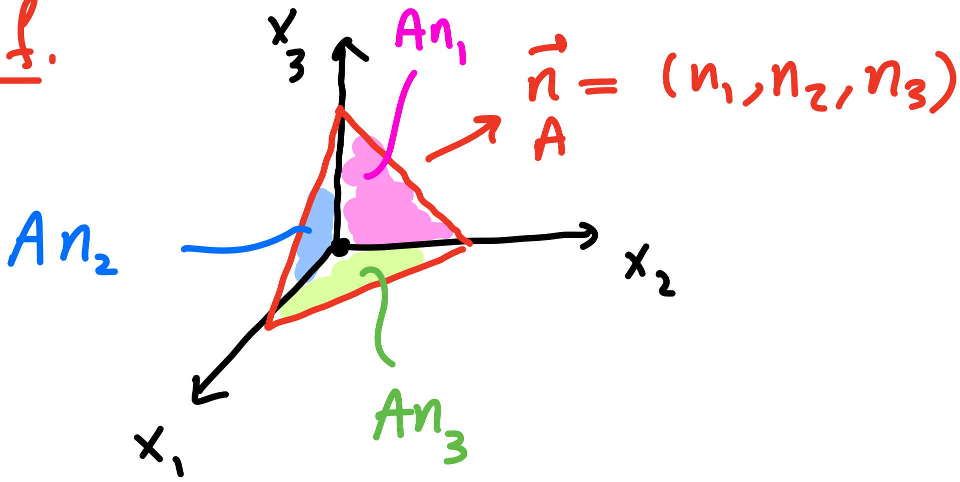
Thm. Cauchy Stress

Traction depends linearly on $\vec{n}$:

$$t(n) = \sigma n \qquad t_i = \sigma_{ij} n_j$$

exists (Cauchy Stress)

Pf.



$$\overset{O(h^2)}{}$$

$$-t(e_1)\cancel{A}n_1 - t(e_2)\cancel{A}n_2 - t(e_3)\cancel{A}n_3$$

$$+\, t(n)\cancel{A} + \underbrace{\cancel{\vec{b}\rho v}}_{o(h^3)} = \overbrace{\cancel{\rho v}}^{o(h^3)} \underset{\text{acceleration}}{a}$$

$$\left\{\begin{aligned} &A \propto h^2 \\ &v \propto h^3 \end{aligned}\right.$$

so, $h \to 0$

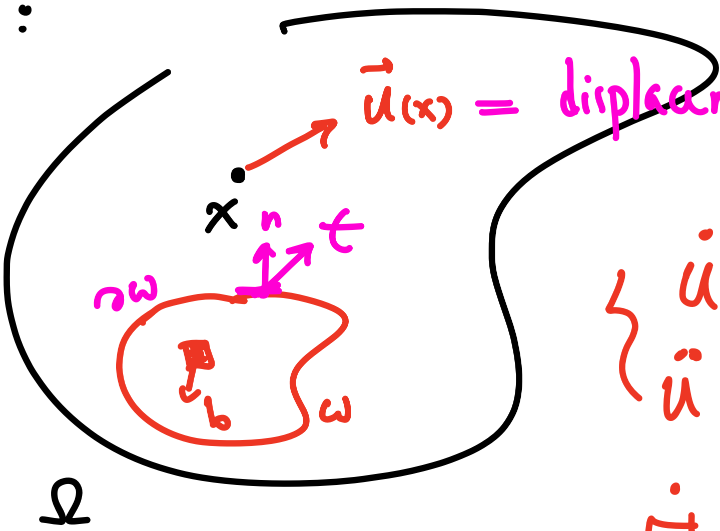
$$t(n) = \underbrace{t(e_i)}_{\sigma} n_i$$

∎

3. Balance laws

Thm. Balance of linear momentum $\iff \nabla \cdot \sigma + b = \rho \ddot{u}$

side note:



$\vec{u}(x) =$ displacement field

$$\begin{cases} \dot{u} = \text{velocity} \\ \ddot{u} = \text{acceleration} \end{cases}$$

$$\dot{\square} = \frac{\partial}{\partial t}$$

pf. Pick subbody $\omega \subseteq \Omega$, then:

$$\cancel{\frac{d}{dt}} \int_{\omega} \rho \ddot{u} \, dV = \int_{\partial\omega} \underset{\substack{\parallel \\ \sigma n}}{t} \, dA + \int_{\omega} b \, dV \quad \Longleftarrow$$

$\rho \, dV = \text{Const.}$ (due to Cauchy Thm)

$$\int_{\omega} \rho \ddot{u} \, dV = \underbrace{\int_{\partial\omega} \sigma n \, dA}_{\substack{\parallel \\ \int_{\omega} \nabla \cdot \sigma \, dV}} + \int_{\omega} b \, dV$$

$$\to \int_\omega \underbrace{(\rho\ddot{u} - \nabla\cdot\sigma - b)}_{=0}\, dV = 0 \qquad \forall \omega \subseteq \Omega$$

Thm. Balance of angular momentum

$$\iff \boxed{\sigma = \sigma^T}$$

pf. Pick $\omega \subseteq \Omega$

$$\cancel{\frac{d}{dt}} \int_\omega x \times \rho\ddot{u}\, dV = \overbrace{\int_{\partial\omega} x \times t\, dA}^{\star} + \int_\omega x \times b\, dV$$

$t = \sigma n$

$$\frac{d}{dt}(x \times \dot{u}) = \cancel{\dot{u} \times \dot{u}}_{0} + \underline{x \times \ddot{u}} \qquad \dot{x} = \dot{u}$$

$$[x \times (\sigma n)]_i = \epsilon_{ijk}\, x_j\, \underbrace{(\sigma n)_k}_{=\sigma_{k\ell} n_\ell} = \ldots$$

$$\star = \int_\omega x \times \nabla\cdot\sigma + \underbrace{\int_\omega \epsilon_{ijk}\, \underline{\sigma_{kj}}}_{=0} \qquad \forall \omega$$

Corollary: Because $\sigma = \sigma^T \Longrightarrow$

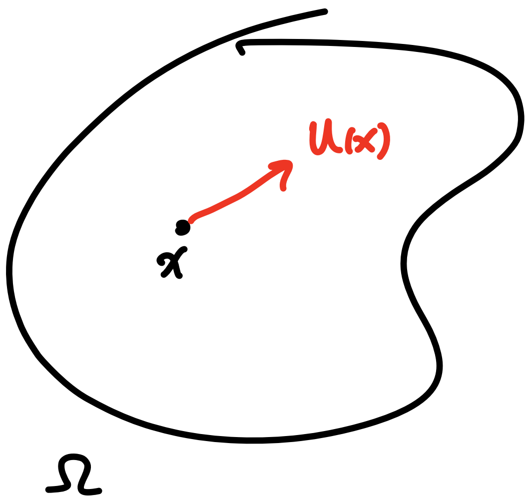
$$\sigma = \sum_{i=1}^{3} \sigma_i \, n_i \otimes n_i$$

spectral decomp.

$$A = A^T \longrightarrow A = P \Lambda P^T \longrightarrow A = \sum_{i=1}^{3} \lambda_i u_i^T u_i$$

$$P = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$$

4. Strain



Definition:

$$\epsilon = \nabla_s u = \frac{\nabla u + \nabla u^T}{2} = \operatorname{sym} \nabla u$$

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\nabla u = \underline{\underline{\epsilon}} + \omega$$

Strain

$$\omega = \frac{\nabla u - \nabla u^T}{2}$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

responsible for solid-body rotations $\neq$ deformation

Definition: Volumetric–Deviatoric split

$$\begin{cases} \sigma = \underbrace{\sigma_m}_{\text{volumetric}} + \underbrace{s}_{\text{deviatoric}} & \text{Stress} \\ \epsilon = \underbrace{\epsilon_v}_{\text{volumetric}} + \underbrace{e}_{\text{deviatoric}} & \text{Strain} \end{cases}$$

$$\sigma_m \equiv \frac{tr(\sigma)}{3} \mathbb{1}$$

"pressure"

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \quad \underline{\underline{2D}}$$

$$tr(\sigma) = \sigma_{11} + \sigma_{22}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$s \equiv \sigma - \sigma_m$$

$$\epsilon_v \equiv tr(\epsilon)\, \mathbb{1}$$

$$//$$

$$\nabla \cdot u = \begin{cases} + & \text{Expansion} \\ - & \text{Contraction} \end{cases} \qquad e = \left(\epsilon_{11} + \epsilon_{22} \right)$$

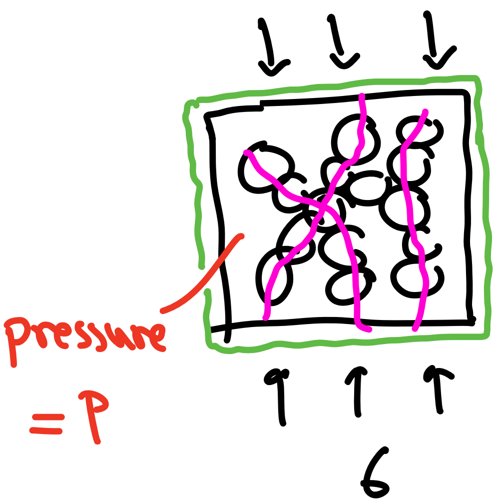
$$e \equiv \epsilon - \epsilon_v$$

$$\begin{cases} \epsilon_{11} = \dfrac{\partial u_1}{\partial x_1} \\ \epsilon_{22} = \dfrac{\partial u_2}{\partial x_2} \end{cases}$$

Remarks :

① Sign Convention

$$\sigma_m = \begin{cases} + & \text{tension} \\ - & \text{Compression} \end{cases}$$



$$\underbrace{\sigma}_{\text{total stress}} = \underbrace{\sigma'}_{\substack{\text{effective stress} \\ \text{(Solid)}}} - \underbrace{b}_{\substack{\text{Biot Coefficient} \\ \text{(=1 soils Terzagi)}}} \underbrace{P}_{\text{pore pressure}} \mathbf{1}$$

b/c $P > 0$

<u>Ex</u>: $\sigma = -100$ Pa

$\sigma' = -60$ Pa

$\longrightarrow P = 40$ Pa

② Specifying $u(x) \longrightarrow \epsilon(x) = \nabla_s u$ is well-defined

Specifying $\epsilon(x) \longrightarrow u(x) \stackrel{!}{=} \int \epsilon(x)$ not well-defined

$\parallel$

$$\begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ & \epsilon_{22} & \epsilon_{23} \\ & & \epsilon_{33} \end{pmatrix}$$

6 Components

overdetermined

$\Longrightarrow$ need constraints

Saint-Venant Constraints

$$\nabla \times (\nabla \times \epsilon) = 0$$

5. linear Elasticity

__Define__. A material is hyperelastic iff:

$$\sigma = \frac{\partial \Psi}{\partial \epsilon}$$

Ψ = elastic energy density / volume.

__Ex__: linear elasticity $\Psi = \frac{1}{2}\, \epsilon : \mathbb{C} : \epsilon$

stiffness tensor (4th order)

$$\Psi = \frac{1}{2}\epsilon_{ij}\,\mathbb{C}_{ijkl}\,\epsilon_{kl} \longrightarrow \frac{\partial\Psi}{\partial\epsilon} = \mathbb{C}:\epsilon = \sigma$$

$$\sigma_{ij} = \frac{\partial\Psi}{\partial\epsilon_{ij}} \qquad \frac{\partial\Psi}{\partial\epsilon_{ij}} = \cdots \frac{\partial\epsilon_{kl}}{\partial\epsilon_{ij}}$$

$$\frac{\partial\epsilon_{kl}}{\partial\epsilon_{ij}} = \delta_{ki}\,\delta_{lj}$$

proposition. Minor Symmetry

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}$$

pf. $\sigma = \sigma^T$, $\epsilon = \epsilon^T$

$$\sigma_{ij} = \sigma_{ji} \longrightarrow \mathbb{C}_{ijkl}\,\epsilon_{kl} = \mathbb{C}_{jikl}\,\epsilon_{kl} \quad \forall \epsilon_{kl}$$

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl}$$

proposition. Major Symmetry

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} \iff \text{linear (hyper)elastic}$$

$$\sigma = \frac{\partial\Psi}{\partial\epsilon} = \mathbb{C}:\epsilon$$

Pf. $(\Leftarrow)$ $\cancel{\Psi = \frac{1}{2} \epsilon : \mathbb{C} : \epsilon}$ $\longrightarrow$ $\boxed{\sigma = \mathbb{C} : \epsilon}$ ✓

$$= \frac{\partial \Psi}{\partial \epsilon}$$

$$\rightarrow \mathbb{C}_{ijkl} = \frac{\partial^2 \Psi}{\partial \epsilon_{ij} \, \partial \epsilon_{kl}} = \mathbb{C}_{klij}$$

$(\Rightarrow)$ $\Psi = \frac{1}{2} \epsilon : \Psi : \epsilon$

$$\frac{\partial \Psi}{\partial \epsilon_{ij}} = \frac{1}{2} \left(\underline{\mathbb{C}_{ijkl}} + \underline{\mathbb{C}_{klij}} \right) \epsilon_{kl} \Rightarrow \sigma$$

$$=$$

✓

Proposition. **Parameter Count of $\mathbb{C}$**

$\mathbb{C}_{ijkl}$ has at most $\underline{\underline{21}}$ constants.

$$3 \times 3 \times 3 \times 3 = 3^4 = 81$$

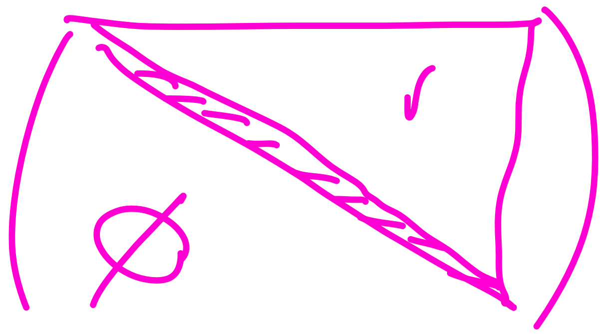
$$ij = \begin{pmatrix} 1,1 & 1,2 & 1,3 \\ & 2,2 & 2,3 \\ & & 3,3 \end{pmatrix}$$

Pf.

$$\mathbb{C}_{\underbrace{ij}_{6}\underbrace{kl}_{6}}$$

unique

pairs

$(ij)\,(kl) =$ 

6×6

$$\frac{6^2-6}{2}+6 = 21$$

linear isotropic stiffness :

$$\mathbb{C}_{ijkl} = \underline{\underline{\lambda}}\,\delta_{ij}\delta_{kl} + \underline{\underline{\mu}}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)$$

Bulk ?   shear

Lamé Constants

END of Slot 1

Variational Calculus

1. Functionals & 1st variations

function $\quad x \longmapsto f(x) \qquad \mathbb{R}\longrightarrow\mathbb{R}$

$\mathbb{C}\longrightarrow\mathbb{C}$

functional $f(x) \longmapsto \mathbb{R}$ or $\cancel{\mathbb{C}}$

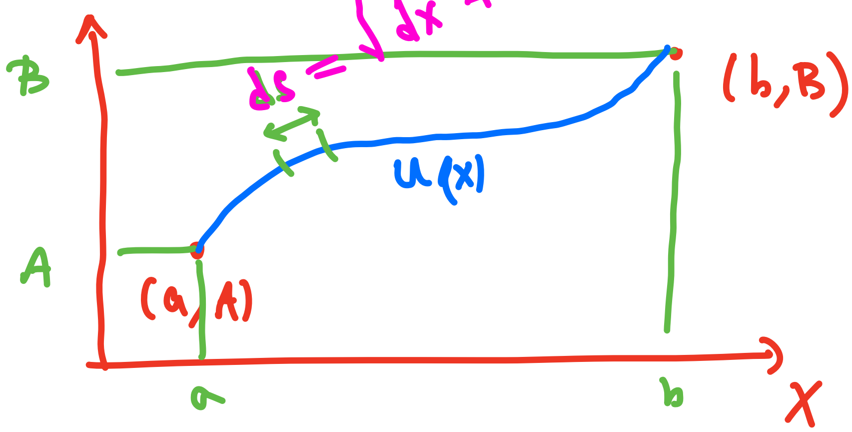
<u>Define</u>.

$$J[u] = \int_0^L F(x, u, u')\, dx$$

$$\left(u(x) \longmapsto \mathbb{R} \right)$$

<u>Ex</u>:

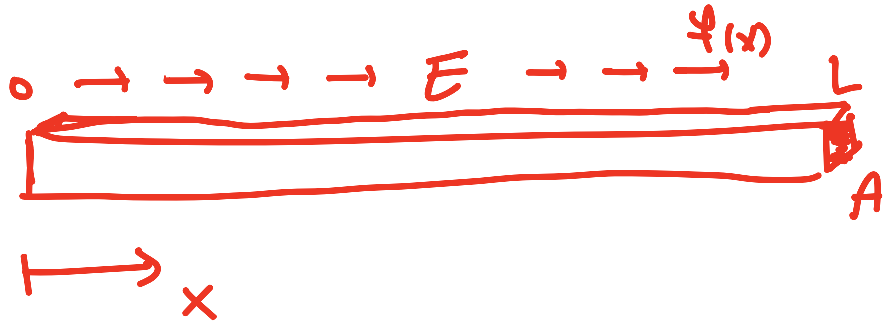
$$ds = \sqrt{dx^2 + du^2} = \sqrt{1 + u'^2}\, dx$$



$$L[u] = \int_0^L ds = \int_0^L \underbrace{\sqrt{1 + u'^2}}_{F(\cancel{x}, \cancel{u}, u')}\, dx$$

<u>Ex</u>:

$$\Pi[u] = \int_0^L \underbrace{(\text{Elastic potential}) - (\text{virtual work done})}_{\frac{1}{2} EA u'^2 - f u}$$



$$\epsilon = \frac{du}{dx} = u' \qquad \sigma = E\epsilon = Eu'$$

$$\Psi = \frac{1}{2}\epsilon : \mathbb{C} : \epsilon = \frac{1}{2} E u'^2$$

$$\Pi[u] = \int_0^L (\Psi A - f u)\, dx$$

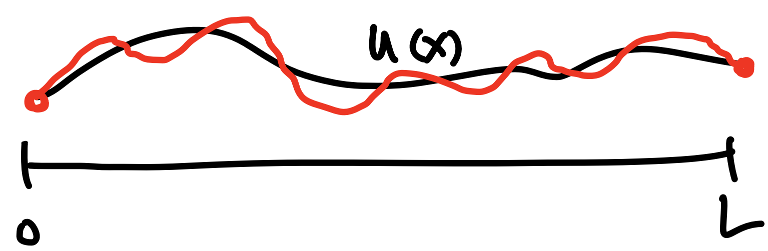
Def. <u>First Variation</u>

$$\delta J \equiv \frac{d}{d\epsilon} J[u + \underbrace{\epsilon\eta}_{\text{perturbation}}] \Big|_{\epsilon = 0}$$

$\eta(x)$ function (Pick / smooth)

ϵ small number

Note: The picked $\eta(x)$ must be admissible



$$\underline{\text{BCs}}\begin{cases} u(0) = u_0 \\ \cancel{u(L) = u_L} \end{cases} \longrightarrow \tilde{u} = u + \epsilon\eta$$

$$\eta(0) = \cancel{\eta(L)} = 0$$

<u>Thm</u>. The variation of J is :

$$\boxed{\delta J = \int_0^L \left(F_u - \frac{d}{dx}F_{u'}\right)\eta + F_{u'}\eta\Big|_0^L}$$

$$\begin{cases} F_u = \dfrac{\partial F}{\partial u} \\ F_{u'} = \dfrac{\partial F}{\partial u'} \end{cases}$$

<u>pf.</u>

$$\delta J = \frac{d}{d\epsilon} J[u+\epsilon\eta]\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\int_0^L F(x, u+\epsilon\eta, u'+\epsilon\eta')\,dx\,\Big|_{\epsilon=0}$$

$$= \int_0^L F_u \eta + \int_0^L F_{u'} \eta'$$

$$\int_0^L F_{u'} \eta' = F_{u'} \eta \Big|_0^L - \int_0^L \frac{d}{dx} F_{u'} \eta$$

$\square$

2. Euler-Lagrange Equations

Lemma. Fundamental lemma

if $g(x)$ is continuous and $\int_0^L g\eta = 0 \ \forall \eta$ (η sufficiently smooth) $\eta(0) = \eta(L) = 0$

$$\Longrightarrow g = 0 \text{ in } x \in [0, L]$$

Thm. Euler-Lagrange Equations

$$\delta J = 0 \quad \forall \eta \qquad \Longrightarrow \qquad \boxed{F_u - \frac{d}{dx} F_{u'} = 0}$$

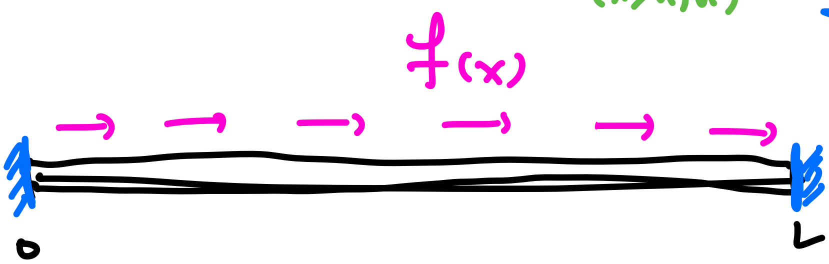
$\eta(0) = \eta(L) = 0$

Ex: Axially loaded bar

$$\Pi[u] = \int_0^L \underbrace{\left(\frac{1}{2}EAu'^2 - fu\right)}_{= F(x,u,u')} dx$$

$\underline{u(0) = u(L) = 0}$

BCs



$\underline{\eta(0) = \eta(L) = 0}$

$$0 = \delta\Pi = \underbrace{F_u}_{= -f} - \frac{d}{dx}\underbrace{F_{u'}}_{= EAu'} = -(EAu')' - f$$

$$= \boxed{-EAu'' - f = 0}$$

if E, A constant

$$\nabla \cdot \sigma + f = 0$$

Question: Does a BVP have a functional "parent"? Not always, must be compatible with BCs

δ- rule :

- δ commutes w/ $\dot{\square}$ and $\frac{\partial}{\partial x}$ $(\delta\epsilon = \nabla_s \delta u)$
- Chain rule applies : $\delta F(u,u') = F_u \delta u + F_{u'} \delta u'$

  NB. $\delta u = \cancel{\epsilon}\eta$

- δ(fixed function) $= 0$ (Ex: $\delta b = 0$)

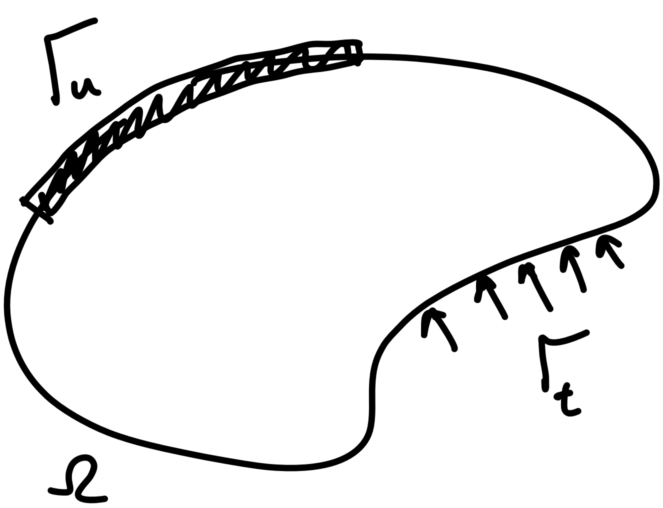
  $\delta \delta u = 0$

  η = fixed

3. Essential and natural BCs

- Essential $\equiv$ Dirichlet BCs

  $\Longrightarrow$ imposed on the admissible set of variations



$\partial\Omega = \Gamma_u \cup \Gamma_t$

Admissible $\eta(x)$

$\eta(x) = 0 \quad \forall x \in \Gamma_u$

- Natural BCs $\Longrightarrow$ show up in the functional itself

<u>Proposition</u> :

$$J[u] = \int_0^L F(x, u, u')\, dx$$

$u(x=0) = \bar{u} \longrightarrow \eta(0) = 0$

at $x=L$ free $\longrightarrow$ BC imposed here ?

must be compatible

$$\rightarrow \delta J = \int_0^L \left(F_u - \frac{d}{dx} F_{u'}\right)\eta + F_{u'}\,\eta \Big|_{x=L}$$

if $\delta J = 0 \Longrightarrow \begin{cases} F_u - \dfrac{d}{dx} F_{u'} = 0 \\ F_{u'} \Big|_{x=L} = 0 \end{cases}$

Boundary Cond. at $x=L$!!

① Pf. First pick η s.t. $\eta(0) = \eta(L) = 0$

$\Longrightarrow$ boundary term $= 0$

$$\boxed{F_u - \frac{d}{dx} F_{u'} = 0} \quad \star$$

② having shown $\star$, Pick now η that is free at $x = L$

from $\star \longrightarrow \delta J = F_{u'} \underbrace{\eta}_{\text{arbitrary}} \Big|_{x=L} \longrightarrow \boxed{F_{u'} \Big|_{x=L} = 0}$

Ex: Consider the BC: $\boxed{F_{u'}(x=L) = \underbrace{P}_{\text{constant force}} \neq 0}$ specify force

WANT

$F_{u'} = EAu'$ force

$\longrightarrow \quad EAu' = P \longrightarrow u' = \#$ Neumann $\equiv$ natural

This BC clashes/incompatible with

$$\delta J = 0 \quad \left(J = \int_0^L F \, dx \right)$$

Q: BVP $\left\{ \begin{array}{l} EAu'' + f = 0 \\ u(x=0) = \bar{u} \\ F_{u'}(x=L) = P \neq 0 \end{array} \right. \longrightarrow$ functional "parent"

compatible

if $\boxed{\hat{J}[u] \equiv \int_0^L F\,dx - P u_L}$ $\Longrightarrow \delta \hat{J} = 0$

"parent"

pf.

$$\delta \hat{J} = \int_0^L \underset{\displaystyle F_u \delta u + F_{u'} \underline{\delta u'}}{\underset{\|}{\delta F}}\, dx - P\,\delta u_L$$

$$0 = \delta \hat{J} = \int_0^L \underbrace{\left(F_u - \frac{d}{dx} F_{u'}\right)}_{=0} \underline{\delta u} + \underbrace{\left(F_{u'} - P\right)}_{=0} \delta u_L$$

Thm. <u>Non-Dirichlet BCs</u>

$$J[u] = \int_0^L F\,dx + B(u_L) \qquad u(0) = \bar{u}$$

Then

$$\delta J = 0 \implies \left\{ \text{E-L eq.} \quad \boxed{F_{u'}\Big|_{x=L} + \underset{-P}{\underline{B'(u_L)}} = 0} \right.$$

Ex.

$$B(u_L) = -Pu_L$$

4. Functionals w/ higher derivative

$$J[w] = \int_0^L \underline{F(x, w, w', w'')}\, dx \qquad x \in [0, L]$$

$$\underline{\delta J = 0} \implies \boxed{F_w - \frac{d}{\underline{dx}} F_{w'} + \frac{d^2}{\underline{dx^2}} F_{w''} = 0}$$

$$\begin{cases} ① \quad w = \bar{w} \quad \text{or} \quad F_{w'} - (F_{w''})' = 0 \\ ② \quad w' = \bar{w}' \quad \text{or} \quad F_{w''} = 0 \end{cases}$$

Pf.

$$\delta J = \int_0^L \delta F$$

$$\delta F = F_w \delta w + F_{w'} \delta w' + F_{w''} \delta w''$$

$\times 1$ $\quad$ $\times 2$

$-$ $\quad$ $+$